

# Constraints on Superfluid Hydrodynamics from Equilibrium Partition Functions

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**ABSTRACT:** Following up on recent work in the context of ordinary fluids, we study the equilibrium partition function of a 3+1 dimensional superfluid on an arbitrary stationary background spacetime, and with arbitrary stationary background gauge fields, in the long wavelength expansion. We argue that this partition function is generated by a 3 dimensional Euclidean effective action for the massless Goldstone field. We parameterize the general form of this action at first order in the derivative expansion. We demonstrate that the constitutive relations of relativistic superfluid hydrodynamics are significantly constrained by the requirement of consistency with such an effective action. At first order in the derivative expansion we demonstrate that the resultant constraints on constitutive relations coincide precisely with the equalities between hydrodynamical transport coefficients recently derived from the second law of thermodynamics.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Equilibrium effective action for the Goldstone mode</b>	<b>5</b>
2.1 The question addressed	5
2.2 The partition function for charged (non super) fluids	5
2.3 Euclidean action for the Goldstone mode for superfluids	6
2.4 The Goldstone action for perfect superfluid hydrodynamics	7
2.5 The Goldstone Action at first order in derivatives	9
<b>3. Constraints on parity even corrections to constitutive relations at first order</b>	<b>14</b>
3.1 Constraints from the local second law	14
3.2 Constraints from the partition function	21
3.3 Entropy from the partition function	25
3.4 Consistency with field redefinitions	29
<b>4. Constraints on parity violating constitutive relations at first order</b>	<b>32</b>
4.1 Review of constraints from the second law	32
4.2 Constraints on constitutive relations from the Goldstone action	36
4.3 Entropy	38
<b>5. CPT Invariance</b>	<b>41</b>
<b>6. Discussion</b>	<b>42</b>

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## 1. Introduction

Hydrodynamics is the long distance effective description of locally thermalized systems. The variables of hydrodynamics are local values of the temperature, chemical potential, velocity and other relevant thermodynamical order parameters. The equations of hydrodynamics are the universal laws of conservation of the stress tensor and the charge current. These equations may be used to describe the propagation of a fluid in an arbitrary weakly curved background metric, and with arbitrary slowly varying

background gauge fields. Within the hydrodynamical approximation, the stress tensor and charge current of the fluid are expressed as functions of thermodynamical and background fields; the formulas through which this is achieved are referred to as the constitutive relations of the hydrodynamical system.

The hydrodynamical constitutive relations of any given system may, in principle, be determined by a detailed study of the dynamics of the theory. For strongly coupled quantum field theories, however, the required calculations usually cannot be practically executed.<sup>1</sup> Given this state of affairs, it is clearly of interest to have a complete parameterization of the most general hydrodynamical constitutive relations allowed on general grounds. Such a characterization would constitute a satisfactory framework for the theory classical hydrodynamics viewed as an autonomous long wavelength effective theory.

The constitutive relations of relativistic hydrodynamics are specified in an expansion in derivatives of the local thermodynamical fields and background fields. At any order in this expansion, Lorentz invariance determines the constitutive relations up to a finite number of functions of the scalar thermodynamical fields (e.g. local temperature and chemical potential). It turns out, however, that other considerations further constrain the constitutive relations. These constraints are of two sorts. The first, and more important, set of constraints asserts relations between the apparently independent functions that appear in the most general Lorentz allowed constitutive relations. Such constraints cut down the number of free functions in the equations of hydrodynamics (at any order in the derivative expansion). We refer to constraints of this first sort as ‘equality type’ constraints; they are the primary focus of the current paper. A second, milder form of constraints assert inequalities for the free functions that appear in constitutive relations. These constraints do not reduce the number of free functions in constitutive relations but merely bound these functions. In this paper we will have nothing to say about this second class of constraints.

One method for obtaining constraints on constitutive relations was outlined in the classic text book of Landau and Lifshitz [4] and refined in later studies ( see e.g. [5, 6, 7, 8, 9, 10] for recent work). This method is based on the assumption that consistent equations of hydrodynamics come equipped with an entropy current. Like the conserved currents, the entropy current is a function of the local thermodynamical fields. The key dynamical assumption is that the divergence of this entropy current is point wise (in spacetime) positive semi definite for every conceivable fluid flow. We refer to the existence of such a positive divergence entropy current - without making

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<sup>1</sup>The only exceptions that we are aware of lie within the fluid gravity map ( [1], see [2, 3] for reviews) of the AdS/CFT correspondence of string theory. The constitutive relations of field theories with a known dual description are rather easily determined following the procedure first described in [1].

any prior assumptions as to its functional form - as a local version of the second law of thermodynamics. The local second law clearly guarantees that net thermodynamical entropy increases in any fluid flow that starts and ends in equilibrium. The requirement that entropy increase for fluid flows perturbed by arbitrary local sources suggests that the local form of the second law is also a necessary consequence of the second law.<sup>2</sup>

At low orders in the derivative expansion it has been demonstrated that the local version of the second law of thermodynamics yields powerful constraints (of both the equality and inequality sort) on otherwise unrelated free functions in constitutive relations. The detailed form of these constraints has been worked out for uncharged relativistic fluids up to second order in the derivative expansion [10] and for (parity non preserving) charged fluids to first order in the derivative expansion (see [5, 11, 6] for 3+1 dimensional fluids and [9] for 2+1 dimensional fluids).

Very recently, a second systematic method for constraining the constitutive relations for fluids was described in [12, 13]<sup>3</sup>. These constraints follow from the very reasonable demand that the hydrodynamical equations must always admit equilibrium solutions for arbitrary stationary background field configurations, and moreover that the values of conserved charges in equilibrium must be consistent with the existence of an equilibrium partition function. For both uncharged fluids at second order as well as charged parity non invariant (and potentially anomalous) fluids at first order, the equality type constraints obtained from this method agree, in full detail [12, 13] with those obtained from the local form of the second law of thermodynamics described above. It has been conjectured [12] that this agreement persists to all orders in the derivative expansion; however there is as yet no proof of this conjecture.

The equations of charged hydrodynamics are modified when the charge symmetry of the system is spontaneously broken by the condensation of a charged operator in thermal equilibrium. The effective description of such systems has new hydrodynamical degrees of freedom whose origin lies in the Goldstone mode of the charge condensate. The resultant hydrodynamical equations are referred to as the equations of superfluid hydrodynamics, and are the subject of the current paper.

More particularly in this paper we study ‘s’ wave superfluid hydrodynamics, i.e. the hydrodynamics of a system whose charge condensate is a complex scalar operator. We study the constraints on the equations of first order ‘s’ wave superfluid hydrodynamics imposed by the requirement that these equations admit equilibrium under

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<sup>2</sup>It is at least suggestive that the fluid dynamics generated by the fluid gravity map of the AdS/CFT correspondence does indeed always obey the local form of the second law, at least when the dual bulk description is given by the equations of two derivative Einstein gravity with matter that obeys the null energy conditions.

<sup>3</sup>See also [14, 15] for earlier related work, [16] for closely related discussions and [17, 18] for follow up work.

appropriate situations, and that the charge currents in equilibrium agree with those from an appropriate partition function. We do not assume that the superfluids we study necessarily preserve either parity or time reversal invariance.

As we explain in section 2 below, the general analysis presented in this paper closely follows that of [12] (for the case of ordinary, i.e. not ‘super’ fluids) with one important difference. The Euclidean partition function for a superfluid in an arbitrary background <sup>4</sup> is determined by an effective field theory that includes a massless mode: the Goldstone boson of the theory. This effective field theory is local, and may usefully be studied in the derivative expansion. However the partition function that follows after integrating out the Goldstone boson is neither local nor simple. As we explain below, the study of the local effective action of the Goldstone boson (rather than the partition function itself) allows us to usefully constrain the constitutive relations of superfluid hydrodynamics. In this paper we present a careful derivation of the relations between otherwise independent transport functions that follow from such a study.

Constraints on the constitutive relations of first order superfluid hydrodynamics have previously been obtained using the local form of the second law in [20, 7, 6, 8] for the case of time reversal invariant superfluids. In this paper we generalize the derivation of [6] to include the study of superfluids that do not preserve time reversal invariance. We then compare the results obtained from the two different methods; i.e. the constraints that follow from the requirement of existence of equilibrium and those that follow from the local second law. As in the case of ordinary (i.e. non super) fluids we find perfect agreement between the equality type constraints obtained from these two apparently distinct methods. Our results supply further evidence for the conjecture that the equality type constraints from these two methods agree in a wide range of hydrodynamical contexts and to all orders in the derivative expansion. A proof of this conjecture would go some way towards proving the local form of the second law, and would permit the demystification of this law in a hydrodynamical context.

While the work reported in this paper is purely hydrodynamical and nowhere uses AdS/CFT, much of the motivation for this work lies within the fluid gravity map of AdS/CFT. The status of the second law of thermodynamics for theories of gravity that include higher derivative corrections to the Einstein Lagrangian remains unclear. In particular it has never been proved that the Hawking area increase theorem generalizes to a Wald entropy increase theorem for arbitrary higher derivative corrections to Einstein’s gravity. If the interplay between the existence of equilibrium in appropriate circumstances and entropy increase can be proved on general grounds in a hydrodynamical context, then it seems likely that the lessons learnt can be taken over to the study of entropy increase in higher derivative gravity (at least for asymptotically AdS

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<sup>4</sup>See [13, 19] for a discussion of this partition function at the perfect fluid level.

space) via the fluid gravity map. This could lead to a proof of a Wald entropy increase theorem under appropriate conditions on the higher derivative corrections of the gravitational system.

## 2. Equilibrium effective action for the Goldstone mode

### 2.1 The question addressed

In this section we study an  $s$  wave superfluid propagating on the stationary background metric

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = -e^{2\sigma(\vec{x})} (dt + a_i(\vec{x}) dx^i)^2 + g_{ij}(\vec{x}) dx^i dx^j \quad (2.1)$$

and background gauge field

$$A = \mathcal{A}_0(\vec{x}) dx^0 + \mathcal{A}_i(\vec{x}) dx^i \quad (2.2)$$

Below we will often work in terms of the modified gauge fields

$$\begin{aligned} A_i &= \mathcal{A}_i - A_0 a_i \\ A_0 &= \mathcal{A}_0 + \mu_0 \end{aligned} \quad (2.3)$$

All background fields above are assumed to vary slowly; we work in an expansion in derivatives of these fields. We address the following question: what is the most general allowed form of the partition function

$$Z = \text{Tre}^{-\frac{H - \mu_0 Q}{T_0}} \quad (2.4)$$

as a function of the background fields  $\sigma$ ,  $a_i$ ,  $g_{ij}$ ,  $A_0$  and  $A_i$  in a systematic derivative expansion?

### 2.2 The partition function for charged (non super) fluids

The analogous question was studied for the case of an ordinary (non super) charged fluid in [12]. It was demonstrated that to first order in the derivative expansion the most general allowed form of the partition function for an ordinary charged fluid on the background (2.1), (2.2) is given by

$$\begin{aligned} W &= \ln Z = W^0 + W_{inv}^1 + W_{anom}^1 \\ W^0 &= \int \sqrt{g} \frac{e^\sigma}{T_0} P(T_0 e^{-\sigma}, e^{-\sigma} A_0) \\ W_{inv}^1 &= \frac{C_0}{2} \int A dA + \frac{C_1}{2} \int a da + \frac{C_2}{2} \int A da \\ W_{anom}^1 &= \frac{C}{2} \left( \int \frac{A_0}{3} A dA + \frac{A_0^2}{6} A da \right) \end{aligned} \quad (2.5)$$

where  $P(T, \mu)$  is the thermodynamical pressure of the system as a function of its temperature and chemical potential and  $C_0$ ,  $C_1$ ,  $C_2$  and  $C$  are all constants. The constant  $C$  specifies the covariant  $U(1)^3$  anomaly via the equation

$$\partial_\mu \tilde{J}^\mu = -\frac{C}{8} * (F \wedge F) \quad (2.6)$$

The constants  $C_0$ ,  $C_1$  and  $C_2$  do not (yet) have similar interpretations. It was demonstrated that  $C_0 = C_1 = 0$  in any system that respects CPT invariance.

Notice that the result (2.5) for the partition function of an ordinary (non super) fluid is a *local* function of the background sources  $g_{ij}$ ,  $a_i$ ,  $\sigma$ ,  $A_0$  and  $A_i$ . Locality is a direct consequence of the fact that the path integral that computes the partition function (2.4) has a unique hydrodynamical saddle point (as opposed to a moduli space of saddle points). As a consequence the partition function is generically<sup>5</sup> computed by a path integral over an action with no massless fields. It follows that the result is local on length scales large compared to the inverse mass gap in the action (this mass gap is sometimes referred to as a static screening length of the 4 d thermal system)<sup>6</sup>.

### 2.3 Euclidean action for the Goldstone mode for superfluids

Unlike an ordinary charged fluid, the equilibrium configuration of a superfluid in the background (2.1) is not unique. As superfluids break the global  $U(1)$  symmetry, every background admits at least a one parameter set of equilibrium configurations that differ by a constant shift in the phase of the expectation value of the condensed scalar. It follows that the path integral that computes (2.4) has a zero mode (the phase of the scalar condensate). Consequently, the partition function (2.4), is *not* a local function of the background source fields. Instead this partition function is generated by a local three dimensional field theory of the *dynamical* phase field  $\phi$ .

The dynamics of the Goldstone boson in general, governed by a 3d massless quantum field theory. In this paper, however, we focus on field theories in an appropriate large  $N$  limit (such as theories with matrix degrees of freedom in the t' Hooft limit). In such a limit the effective action for the Goldstone boson is multiplied by a suitable positive power of  $N$  (the factor is  $N^2$  in the t'Hooft limit mentioned above). As a consequence Goldstone dynamics is effectively classical in the large  $N$  limit. Quantum corrections to this classical answer, which are suppressed by appropriate powers of  $N$  (this power is  $\frac{1}{N^2}$  in the t'Hooft limit), may have very interesting structure, see e.g. [22, 23, 24, 25] for related work. We leave their study to future work.<sup>7</sup>

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<sup>5</sup>Non hydrodynamical massless modes occur when the system is tuned to a second order phase transition. We assume in what follows that our system has not been tuned to such a phase transition. We leave the study of this interesting special case [21] to future work.

<sup>6</sup>We thank K. Jensen for discussions on this topic

<sup>7</sup>We thank K. Jensen for discussions on this topic.

In principle, the partition function (2.4) for the superfluid may be obtained from the corresponding local effective action by integrating out the Goldstone boson (i.e. solving its equation of motion and plugging the solution back into the action).<sup>8</sup> In practice the implementation of this procedure requires the solution of a nonlinear partial differential equation. Moreover, even if one could solve this equation the resultant partition function would be highly nonlocal. A direct analysis of the partition function itself seems neither easy nor particularly useful. In order to obtain constraints on the equations of superfluid hydrodynamics below we will work directly with the local effective action for the Goldstone mode rather than the final result for the partition function.

The requirements of gauge invariance significantly constrain the form of Goldstone effective action. Let  $\phi$  denote the phase of the scalar condensate. Under a gauge transformation  $\mathcal{A}_i \rightarrow \mathcal{A}_i + \partial_i \alpha$ ,  $\phi$  transforms as  $\phi + \alpha$ . It follows that the effective action can only depend on the combination

$$\xi_i = -\partial_i \phi + \mathcal{A}_i$$

Note that  $\xi_\mu$  like  $\mathcal{A}_\mu$ , is a field of zero order in the derivative expansion<sup>9</sup>.

The local field theory for the Goldstone boson must also enjoy invariance under Kaluza Klein gauge transformations ( $a_i \rightarrow a_i - \partial_i \gamma$ , see subsection 2.2 of [12] for details). For this reason we work with the Kaluza Klein invariant fields

$$\zeta_i = \xi_i - a_i A_0 = -\partial_i \phi + A_i. \quad (2.7)$$

We also define

$$\xi_0 = A_0$$

and define

$$\chi = \xi^2 = -\xi_\mu \xi^\mu = \xi_0^2 e^{-2\sigma} - g^{ij} \zeta_i \zeta_j. \quad (2.8)$$

## 2.4 The Goldstone action for perfect superfluid hydrodynamics

As we have explained above, the euclidean partition function for our system is generated by an effective action  $S$  for the Goldstone field  $\phi$ . This Goldstone action may be expanded in a power series in derivatives.

$$S = S_0 + S_1 + S_2 \dots \quad (2.9)$$

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<sup>8</sup>If the Euclidean 3 dimensional manifold we work on is compact and we demand single valuedness of the field  $\phi$  then it is plausible that the solution to the  $\phi$  equation of motion is (at least generically) unique, see below.

<sup>9</sup>This means that the phase field  $\phi$  is of  $-1$  order in derivatives; this observation does not invalidate the derivative expansion as gauge invariant physical quantities are functions only of  $\xi^\mu$  and not independently of  $\phi$ .



At lowest (zero) order in the derivative expansion symmetries constrain the Goldstone boson effective action to take the form<sup>10</sup>

$$\begin{aligned} S_0 &= \int d^3x \sqrt{g} \frac{1}{\hat{T}} P(\hat{T}, \hat{\mu}, \chi). \\ \hat{T} &= T_0 e^{-\sigma} \\ \hat{\mu} &= A_0 e^{-\sigma} \\ \hat{u}^\mu &= (1, 0, 0, 0) e^{-\sigma} \end{aligned} \tag{2.10}$$

where  $P$  is an arbitrary function whose thermodynamical significance we will soon discover, and  $\chi$  was defined in (2.8). The fields  $\hat{T}$ ,  $\hat{\mu}$  and  $\hat{u}^\mu$  are the values of the hydrodynamical temperature, chemical potential and velocity fields in equilibrium at zeroth order in the derivative expansion (see [12]).

In the classical (or large  $N$ ) limit adopted throughout this paper, the partition function  $Z$  of our system is obtained by evaluating the Goldstone action on shell. Let the solution to the equation of motion be denoted by

$$\zeta_i(x) = \zeta_i^{eq}(x).$$

Then the partition function is given by

$$\ln Z = S(\zeta_i^{eq}(x)) \tag{2.11}$$

At lowest order in the derivative expansion, the action (2.10) depends only on first derivatives of the massless field  $\phi$ . Varying this action w.r.t.  $\phi$

$$\begin{aligned} \delta S_0 &= \int d^3x \sqrt{g} \frac{e^\sigma}{T_0} \frac{\partial P}{\partial \chi} 2g^{ij} \zeta_i \partial_j \delta \phi \\ &= - \int d^3x \frac{1}{T_0} \partial_j (\sqrt{-G} f \zeta^j) \delta \phi \end{aligned} \tag{2.12}$$

yields

$$\partial_j (\sqrt{-G} f \zeta^j) = \nabla_\mu^{(4)} (f \xi^\mu) = \nabla_i \left( \frac{f}{T} \zeta^i \right) = 0. \tag{2.13}$$

where

$$f = 2 \frac{\partial P}{\partial \chi}.$$

Note this equation of motion is of second order in derivatives of the field  $\phi$ .<sup>11</sup> Plugging the solution to (2.13) back into the (2.12) in principle yields an explicit though com-

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<sup>10</sup>The action (2.10) was already presented in [13]. The presentation of this subsection differs from [13] only in the emphasis that  $\phi$  be regarded as a dynamical field in (2.10), rather than a background like  $\hat{T}$ . For related discussions on effective action for superfluid, see for example [26, 19].

<sup>11</sup>The formal similarity of (2.13) to the equation  $\nabla^2 \phi = 0$  (where the Laplacian is taken in an appropriately rescaled metric) suggests that (2.13) has a unique solution on a compact manifold (up to constant shift in  $\phi$ ) provided that  $\phi$  is required to be single valued and smooth on this manifold. However we do not have a proof of this statement.

plicated and nonlocal expression for the partition function of the system as a function of source fields.

The stress tensor and charge current that follow from the action (2.10) may be computed in a straightforward manner using the formulas listed in eqs.(2.16) of [12]; they are given by

$$\begin{aligned}
J_0 &= -\frac{T_0 e^\sigma}{\sqrt{g}} \frac{\delta S_0}{\delta A_0} = -e^{2\sigma} \left[ e^{-\sigma} \frac{\partial P}{\partial \mu} + \frac{\partial P}{\partial \chi} \frac{\partial \chi}{\partial A_0} \right] = -q e^\sigma - \xi_0 f \\
J^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g}} \frac{\delta S_0}{\delta A_i} = \frac{\partial P}{\partial \chi} \frac{\partial \chi}{\partial A_i} = -f \xi^i \\
T_{00} &= -\frac{T_0 e^\sigma}{\sqrt{g}} \frac{\delta S_0}{\delta \sigma} = -e^{2\sigma} \left[ P + \frac{\partial P}{\partial T_0 e^{-\sigma}} \frac{\partial T_0 e^{-\sigma}}{\partial \sigma} + \frac{\partial P}{\partial \mu} \frac{\partial \mu}{\partial \sigma} + \frac{\partial P}{\partial \chi} \frac{\partial \chi}{\partial \sigma} \right] \\
&= -e^{2\sigma} [P - sT - q\mu - f\xi_0^2 e^{-2\sigma}] = e^{2\sigma} \epsilon + f\xi_0^2 \\
T_0^i &= \frac{T_0}{e^\sigma \sqrt{g}} \left[ \frac{\delta S_0}{\delta a_i} - A_0 \frac{\delta S_0}{\delta A_i} \right] = \frac{\partial P}{\partial a_i} - A_0 \frac{\partial P}{\partial A_i} = -A_0 \frac{\partial P}{\partial \chi} \frac{\partial \chi}{\partial A_i} = f A_0 \xi^i \\
T^{ij} &= \frac{-2T_0}{e^\sigma \sqrt{g}} g^{ik} g^{jl} \frac{\delta S_0}{\delta g^{kl}} = -2g^{ik} g^{jl} \left[ -\frac{1}{2} g_{kl} P + \frac{\partial P}{\partial \chi} \frac{\partial \chi}{\partial g^{kl}} \right] = P g^{ij} + f \xi^i \xi^j
\end{aligned} \tag{2.14}$$

The gauge and diffeomorphism invariance of the action (2.10) ensure the stress tensor and charge current described above are automatically conserved onshell (i.e. upon imposing the equation of motion (2.13)).

The complicated looking expressions for the conserved currents (2.14) may actually be summarized in a remarkably simple form as

$$\begin{aligned}
T^{\mu\nu} &= (\epsilon + P) \hat{u}^\mu \hat{u}^\nu + P g^{\mu\nu} + f \xi^\mu \xi^\nu \\
J^\mu &= q \hat{u}^\mu - f \xi^\mu,
\end{aligned} \tag{2.15}$$

where  $\hat{u}$  was defined in (2.10) and all terms on the RHS of (2.15) are evaluated on the zero order equilibrium solutions  $T(x) = \hat{T}$  and  $\mu(x) = \hat{\mu}$ , defined in (2.10) and the functions  $\epsilon$ ,  $s$  and  $q$  are defined in terms of the pressure  $p$  by the equations

$$\begin{aligned}
\epsilon + P &= sT + q\mu \\
dP &= s dT + q d\mu + \frac{1}{2} f d\chi
\end{aligned} \tag{2.16}$$

(2.15) and (3.2) are precisely the Landau-Tisza constitutive relations of superfluid hydrodynamics.

## 2.5 The Goldstone Action at first order in derivatives

One derivative corrections to the Goldstone action (2.10) may be divided into parity even and parity odd terms. We consider these in turn.

### 2.5.1 Parity even one derivative corrections

The most general parity preserving one derivative correction to (2.10) is given by

$$S = S_0 + S_1^{even}$$

$$S_1^{even} = \int d^3y \sqrt{g} \left[ \frac{f_1}{\hat{T}} (\zeta \cdot \partial) \hat{T} + \frac{f_2}{\hat{T}} (\zeta \cdot \partial) \hat{\nu} - f_3 \nabla_i \left( \frac{f}{\hat{T}} \zeta^i \right) \right] \quad (2.17)$$

where  $\hat{T}$  was defined in (2.10),

$$\hat{\nu} = \frac{\hat{\mu}}{\hat{T}} = \frac{A_0}{T_0}$$

and

$$f_i = f_i(\hat{T}, \hat{\nu}, \zeta^2) \quad (i = 1 \dots 3)$$

are arbitrary functions while  $f$  was defined in the previous subsection

$$f(\hat{T}, \hat{\nu}, \zeta^2) = -2 \frac{\partial P}{\partial \zeta^2}$$

Two remarks are in order

- 1. In (2.17) the unspecified function  $f_3$  multiplies the zero order equation of motion of the phase field  $\phi$ . As a consequence, under the field redefinition

$$\phi = \tilde{\phi} + \delta\phi(\hat{T}, \hat{\nu}, \zeta)$$

$$\Rightarrow \xi_\mu = \tilde{\xi}_\mu - \partial_\mu(\delta\phi) \quad (2.18)$$

we find

$$S_0[\phi] = S_0[\tilde{\phi}] - \int d^3x \sqrt{g} \nabla_j \left( \frac{f}{\hat{T}} \zeta^j \right) \delta\phi \quad (2.19)$$

In other words we are free to use the variable  $\tilde{\phi}$  instead of  $\phi$ ; however the first derivative correction with this choice of variable,  $\tilde{S}_1^{even}$ , differs from  $S_1^{even}$  by

$$\tilde{S}_1^{even} = S_1^{even} - \int d^3x \sqrt{g} \nabla_j \left( \frac{f}{\hat{T}} \zeta^j \right) \delta\phi \quad (2.20)$$

In other words the field redefinition (2.18) induces the shifts

$$\tilde{f}_1 - f_1 = 0, \quad \tilde{f}_2 - f_2 = 0, \quad \tilde{f}_3 - f_3 = \delta\phi \quad (2.21)$$

(where  $\tilde{f}_1$ ,  $\tilde{f}_2$  and  $\tilde{f}_3$  are the functions that appear in the expansion of  $\tilde{S}_1^{even}$ , see (2.17) ) For this reason, the dependence of all physical quantities - like the fluid constitutive relations - on  $f_3$  is rather trivial, and easy to deduce on general grounds, as we will see below.

- 2. While the fields  $\sigma$ ,  $\mu$  and  $\chi$  are even under the action of time reversal, the fields  $\xi_i$  and  $\zeta_i$  are odd under this operation. It follows that each of the three terms in (2.18) is odd under the action of time reversal. In other words the simultaneous requirement of parity and time reversal invariance simply sets  $W_1 = 0$ . It follows that time reversal invariant superfluids have no non dissipative transport coefficients at first order.

The corrections from (2.10) to the charge current and stress tensor (2.14) in equilibrium are given by

$$\begin{aligned}
\delta J_0 &= -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}} \left[ \frac{\delta S_1^{even}}{\delta A_0} \right]_{\zeta=\zeta^{eq}} = -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}} \left( \frac{\delta W_1^{even}}{\delta A_0} \right) = -\frac{e^\sigma}{\sqrt{g}} \left( \frac{\delta W_1^{even}}{\delta \hat{\nu}} \right) \\
&= -e^\sigma \left[ \frac{\partial}{\partial \hat{\nu}} \left( \frac{f_1}{\hat{T}} \right) (\zeta^{eq} \cdot \partial) \hat{T} + \frac{\partial}{\partial \hat{\nu}} \left( \frac{f_2}{\hat{T}} \right) (\zeta^{eq} \cdot \partial) \hat{\nu} + \frac{\partial}{\partial \hat{\nu}} \left( \frac{f}{\hat{T}} \right) (\zeta^{eq} \cdot \partial) f_3 - \frac{f}{\hat{T}} (\zeta^{eq} \cdot \partial) \left( \frac{f_2}{f} \right) \right] \\
\delta J^i &= \frac{\hat{T}}{\sqrt{g}} \left( \frac{\delta S_1^{even}}{\delta A_i} \right)_{\zeta=\zeta^{eq}} = \frac{\hat{T}}{\sqrt{g}} \left( \frac{\delta W_1^{even}}{\delta A_i} \right) \\
&= 2(\zeta^{eq})^i \left[ \frac{\partial f_1}{\partial (\zeta^{eq})^2} (\zeta^{eq} \cdot \partial) \hat{T} + \frac{\partial f_2}{\partial (\zeta^{eq})^2} (\zeta^{eq} \cdot \partial) \hat{\nu} + \frac{\partial f}{\partial (\zeta^{eq})^2} (\zeta^{eq} \cdot \partial) f_3 \right] \\
&\quad + g^{ij} \left( f_1 \partial_j \hat{T} + f_2 \partial_j \hat{\nu} + f \partial_j f_3 \right)
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\delta T_{00} &= -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}} \left[ \frac{\delta S_1^{even}}{\delta \sigma} \right]_{\zeta=\zeta^{eq}} = -\frac{\hat{T}e^{2\sigma}}{\sqrt{g}} \left( \frac{\delta W_1^{even}}{\delta \sigma} \right) = \frac{\hat{T}^2 e^{2\sigma}}{\sqrt{g}} \left( \frac{\delta W_1^{even}}{\delta \hat{T}} \right) \\
&= \hat{T}^2 e^{2\sigma} \left[ \frac{\partial}{\partial \hat{T}} \left( \frac{f_1}{\hat{T}} \right) (\zeta^{eq} \cdot \partial) \hat{T} + \frac{\partial}{\partial \hat{T}} \left( \frac{f_2}{\hat{T}} \right) (\zeta^{eq} \cdot \partial) \hat{\nu} \right. \\
&\quad \left. + \frac{\partial}{\partial \hat{T}} \left( \frac{f}{\hat{T}} \right) (\zeta^{eq} \cdot \partial) f_3 - \frac{f}{\hat{T}} (\zeta^{eq} \cdot \partial) \left( \frac{f_1}{f} \right) \right]
\end{aligned} \tag{2.23}$$

$$\delta T_0^i = \frac{\hat{T}}{\sqrt{g}} \left( \frac{\delta S_1^{even}}{\delta a_i} \right)_{\zeta=\zeta^{eq}} = \frac{\hat{T}}{\sqrt{g}} \left( \frac{\delta W_1^{even}}{\delta a_i} \right)_{|A_i=Constant} = -A_0 \delta J^i \tag{2.24}$$

$$\begin{aligned}
\delta T^{ij} &= -\frac{\hat{T}}{\sqrt{g}} g^{il} g^{jm} \left[ \frac{\delta S_1^{even}}{\delta g^{ij}} \right]_{\zeta=\zeta^{eq}} = -\frac{\hat{T}}{\sqrt{g}} g^{il} g^{jm} \left( \frac{\delta W_1^{even}}{\delta g^{ij}} \right) \\
&= -[(\zeta^{eq})^i \delta J^j + (\zeta^{eq})^j \delta J^i] + g^{ij} \left[ f_1 (\zeta^{eq} \cdot \partial) \hat{T} + f_2 (\zeta^{eq} \cdot \partial) \hat{\nu} + f (\zeta^{eq} \cdot \partial) f_3 \right] \\
&\quad + 2(\zeta^{eq})^i (\zeta^{eq})^j \left[ \frac{\partial f_1}{\partial (\zeta^{eq})^2} (\zeta^{eq} \cdot \partial) \hat{T} + \frac{\partial f_2}{\partial (\zeta^{eq})^2} (\zeta^{eq} \cdot \partial) \hat{\nu} + \frac{\partial f}{\partial (\zeta^{eq})^2} (\zeta^{eq} \cdot \partial) f_3 \right]
\end{aligned} \tag{2.25}$$

In equations (2.22) and (2.23) all the scalar functions  $f_1, f_2, f_3$  and  $f$  have been treated as functions  $\hat{T}, \hat{\nu}$  and  $(\zeta^{eq})^2$  respectively. In obtaining (2.22) we have used the zeroth order equation of motion for  $\phi$ .

$$\nabla_i \left( \frac{f}{\hat{T}} (\zeta^{eq})^i \right) = 0$$

to simplify the expressions presented above .

### 2.5.2 Parity violating terms

The most general parity odd contributions to the action are given by<sup>12</sup>

$$\begin{aligned} S^{odd} &= S_1^{odd} + S_{anom} \\ S_1^{odd} &= \int \sqrt{g} d^3x \left( g_1 \epsilon^{ijk} \zeta_i \partial_j A_k + T_0 g_2 \epsilon^{ijk} \zeta_i \partial_j a_k \right) + \frac{C_1}{2} \int ada \\ S_{anom} &= \frac{C}{2} \left( \int \frac{A_0}{3} AdA + \frac{A_0^2}{6} Ada \right) \end{aligned} \quad (2.26)$$

<sup>13</sup> where

$$g_1 = g_1(\hat{T}, \hat{\nu}, \psi), \quad g_2 = g_2(\hat{T}, \hat{\nu}, \psi),$$

$C_1$  is a constant and

$$\hat{\nu} = \frac{\hat{\mu}}{\hat{T}}, \quad \psi = \frac{\zeta^2}{\hat{T}^2}.$$

(We emphasize that we have slightly changed notation compared to the previous subsection. The independent variables for all functions in this subsection are  $\hat{T}, \hat{\nu}$  and  $\psi$ . The corresponding variables in the previous subsection were  $\hat{T}, \hat{\nu}$  and  $\zeta^2$ .)

Note that (2.26) is automatically even under time reversal. The corrections induced by (2.26) to the stress tensor and *consistent* charge current ([27], see section 2.3 equation

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<sup>12</sup>Our convention is  $\frac{1}{2} \int X dY = \int d^3x \sqrt{g_3} \epsilon^{ijk} X_i \partial_j Y_k$ .

<sup>13</sup>The action for parity odd (non super) fluids (2.5) also contains the terms

$$W = \frac{C_0}{2} \int AdA + \frac{C_2}{2} \int Ada.$$

But using the fact that  $\zeta_i = A_i + \partial_i \phi$  and

$$\int \sqrt{g} \epsilon^{ijk} \partial_i \phi \partial_j A_k = 0,$$

we can absorb  $C_0$  in  $g_1$  and  $C_2$  in  $g_2$ .

2.16 of [12]) in equilibrium are given by

$$\begin{aligned}
T^{ij} &= -\frac{2}{\hat{T}}(\zeta^{eq})^i(\zeta^{eq})^j (g_{1,\psi_{eq}}S_1 + T_0g_{2,\psi_{eq}}S_2) \\
T_{00} &= -T_0e^\sigma \left( (-\hat{T}g_{1,\hat{T}} + 2\psi_{eq}g_{1,\psi_{eq}})S_1 + T_0(-\hat{T}g_{2,\sigma} + 2\psi_{eq}g_{2,\psi})S_2 \right) \\
J_0 &= -e^\sigma (g_{1,\nu}S_1 + T_0g_{2,\nu}S_2) - e^\sigma \epsilon^{ijk} \left[ \frac{C}{3}A_i\nabla_j A_k + \frac{C}{3}A_0A_i\nabla_j a_k \right] \\
J^i &= \hat{T} \left( 2g_1(S_1 \frac{(\zeta^{eq})^i}{\hat{T}^2\psi_{eq}} - \frac{V_3^i}{\hat{T}^2\psi_{eq}}) + T_0g_2(S_2 \frac{(\zeta^{eq})^i}{\hat{T}^2\psi_{eq}} - \frac{V_4^i}{\hat{T}^2\psi_{eq}}) + \hat{T}V_1^i g_{1,\hat{T}} - \frac{1}{T_0}V_2^i g_{1,\nu} - V_5^i g_{1,\psi_{eq}} \right) \\
&\quad + \frac{2}{\hat{T}}\zeta^i(S_1g_{1,\psi_{eq}} + T_0S_2g_{2,\psi_{eq}}) \\
&\quad + e^{-\sigma} \left[ 2 \left( \frac{C}{3}A_0 \right) \frac{1}{\hat{T}^2\psi_{eq}} ((\zeta^{eq})^i S_1 - V_3^i) + \left( \frac{C}{6}A_0^2 \right) \frac{1}{\hat{T}^2\psi_{eq}} ((\zeta^{eq})^i S_2 - V_4^i) + \frac{C}{3}\epsilon^{ijk}A_k\nabla_j A_0 \right] \\
T_0^i &= \hat{T} \left( \frac{(T_0g_2 - 2A_0g_1)}{\hat{T}^2\psi_{eq}}(S_1(\zeta^{eq})^i - V_3^i) - \frac{T_0A_0g_2}{\hat{T}^2\psi_{eq}}(S_2(\zeta^{eq})^i - V_4^i) + T_0(\hat{T}V_1^i(g_{2,\hat{T}} - \hat{\nu}g_{1,\hat{T}}) \right. \\
&\quad \left. - \frac{1}{T_0}V_2^i(g_{2,\hat{\nu}} - \hat{\nu}g_{1,\nu}) - V_5^i(g_{2,\psi_{eq}} - \hat{\nu}g_{1,\psi_{eq}})) - \frac{2A_0}{\hat{T}}\zeta^i(S_1g_{1,\psi_{eq}} + T_0S_2g_{2,\psi_{eq}}) \right) \\
&\quad - \frac{1}{2}CA_0^2e^{-\sigma} \left( \frac{1}{\hat{T}^2\psi_{eq}}(\zeta^{eq})^i S_1 - \frac{1}{\hat{T}^2\psi_{eq}}V_3^i \right) + (2C_1 - \frac{C}{6}A_0^3)e^{-\sigma} \left( \frac{1}{\hat{T}^2\psi_{eq}}(\zeta^{eq})^i S_2 - \frac{1}{\hat{T}^2\psi_{eq}}V_4^i \right),
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
\psi_{eq} &= \frac{\zeta_i^{eq}\zeta_j^{eq}g^{ij}}{\hat{T}^2} \\
S_1 &= \epsilon^{ijk}\zeta_i^{eq}\partial_j\zeta_k^{eq}, \quad S_2 = \epsilon^{ijk}\zeta_i^{eq}\partial_j a_k \\
V_1^i &= \epsilon^{ijk}\zeta_j^{eq}\partial_k\sigma, \quad V_2^i = \epsilon^{ijk}\zeta_j^{eq}\partial_k A_0, \quad V_3^i = \epsilon^{ijk}\zeta_j^{eq}F_{kl}(\zeta^{eq})^l \\
V_4^i &= \epsilon^{ijk}\zeta_j^{eq}f_{kl}(\zeta^{eq})^l, \quad V_5^i = \epsilon^{ijk}\zeta_j^{eq}\partial_k\psi_{eq} \\
V_6^i &= \epsilon^{ijk}F_{jk}, \quad V_7^i = \epsilon^{ijk}f_{jk}.
\end{aligned} \tag{2.28}$$

The symbols for  $V_6^i$  and  $V_7^i$  have been introduced for notational convenience only; these vectors are determined in terms of the other quantities above by

$$\begin{aligned}
V_6^i &= \frac{2}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_1 - V_3^i) \\
V_7^i &= \frac{2}{\hat{T}^2\psi_{eq}}((\zeta^{eq})^i S_2 - V_4^i).
\end{aligned} \tag{2.30}$$

As we have emphasized, the formulas above determine the consistent current. The covariant current is obtained from the consistent current by an additional shift (see

section 2.4 of [12] for a review). We find that the one derivative contribution to the covariant current in equilibrium is given by

$$\begin{aligned}
J_0 &= -e^\sigma (g_{1,\hat{\nu}} S_1 + T_0 g_{2,\hat{\nu}} S_2) \\
J^i &= \hat{T} \left( 2g_1 \left( S_1 \frac{(\zeta^{eq})^i}{\hat{T}^2 \psi_{eq}} - \frac{V_3^i}{\hat{T}^2 \psi_{eq}} \right) + T_0 g_2 \left( S_2 \frac{(\zeta^{eq})^i}{\hat{T}^2 \psi_{eq}} - \frac{V_4^i}{\hat{T}^2 \psi_{eq}} \right) + \hat{T} V_1^i g_{1,\hat{T}} - \frac{1}{T_0} V_2^i g_{1,\hat{\nu}} - V_5^i g_{1,\psi_{eq}} \right) \\
&\quad + \frac{2}{\hat{T}} (\zeta^{eq})^i (S_1 g_{1,\psi_{eq}} + T_0 S_2 g_{2,\psi_{eq}}) \\
&\quad + e^{-\sigma} \left[ C \frac{1}{\hat{T}^2 \psi_{eq}} ((\zeta^{eq})^i S_1 - V_3^i) + \left( \frac{C}{2} A_0^2 \right) \frac{1}{\hat{T}^2 \psi_{eq}} ((\zeta^{eq})^i S_2 - V_4^i) \right]
\end{aligned} \tag{2.31}$$

### 3. Constraints on parity even corrections to constitutive relations at first order

In this subsection we will determine parity even first order corrections to the superfluid constitutive relations both from the method of entropy increase as well as from the partition function of the previous section, and demonstrate their equality.

Let us first consider the almost trivial case of parity even superfluids that also preserve time reversal invariance. As we have explained in the previous section, in this case  $W_1 = 0$ . It follows immediately from this result that all non dissipative superfluid transport coefficients must vanish. Exactly this conclusion was reached in [6] from the requirement of point wise positivity of the divergence of the entropy current in an arbitrary fluid flow.

The study of time reversal non invariant superfluids is more involved. In this case the constraints from the local second law have not previously been analyzed. In this section we first present this analysis. We then study the constraints obtained from the analysis of the partition function. As mentioned above, we will find that these two methods yield identical constraints.

#### 3.1 Constraints from the local second law

In this subsection (but nowhere else in this paper) we consider the non equilibrium flow of a superfluid on a (generically) non stationary spacetime. We continue to denote the background metric of our spacetime by  $G_{\mu\nu}$ . The background gauge field is denoted by  $\mathcal{A}_\mu$ . The variables of superfluid hydrodynamics are the temperature field  $T(x^\mu)$ , velocity field  $u^\mu(x^\mu)$  and the gradient of the phase field  $\xi_\mu = -\partial_\mu \phi + \mathcal{A}_\mu$ . We often work in terms of the fluid dynamical field

$$(\zeta_f)_\mu = \xi_\mu + \mu u_\mu$$

Note that, in equilibrium and at lowest order in the derivative expansion  $(\zeta_f)_0 = 0$  and

$$(\zeta_f)_i = \xi_i - \mathcal{A}_0 a_i = \zeta_i.$$

We specify some additional notation that we will use extensively below.

$$\begin{aligned} P^{\mu\nu} &= u^\mu u^\nu + G^{\mu\nu}, \quad \tilde{P}^{\mu\nu} = P^{\mu\nu} - \frac{(\zeta_f)^\mu (\zeta_f)^\nu}{(\zeta_f)^2}, \quad V^\mu = \frac{E^\mu}{T} - P^{\mu\nu} \partial_\nu \nu \\ R &= \frac{q}{\epsilon + P}, \quad K = \nabla_\mu (f \xi^\mu) = s(u \cdot \partial) \left( \frac{q}{s} \right), \quad \Theta = (\nabla \cdot u) = -\frac{u \cdot \partial s}{s} \\ \mathbf{a}_\mu &= (u \cdot \nabla) u_\mu \\ H_1 &= T, \quad H_2 = \nu, \quad H_3 = (\zeta_f)^2 \end{aligned} \quad (3.1)$$

In words,  $P^{\mu\nu}$  projects onto the three dimensional subspace orthogonal to the normal fluid, while  $\tilde{P}^{\mu\nu}$  projects onto the two dimensional subspace orthogonal to both the normal and superfluid velocities.  $\mathbf{a}_\mu$  and  $\Theta$  are the normal fluid acceleration and expansion respectively.  $V^\mu$  is the ‘Einstein combination’ of the electric field and derivative of the chemical potential that vanishes in equilibrium.  $H_1$ ,  $H_2$  and  $H_3$  are new names for the three scalar hydrodynamical fields; note that  $H_2$  is  $\nu = \frac{\mu}{T}$  rather than the chemical potential itself. Finally  $K$  is the term that is set to zero by the first order equation of motion of the Goldstone phase, while  $R$  is a combination of zero order thermodynamical fields that often appears in the formulas below.

In order to analyze the constraints that follow from the local form of the second law, we follow the procedure described in section 3 of [6]. Briefly, we first write down the most general onshell independent first order entropy current allowed by symmetry. We then compute the divergence of this current (this is mere algebra) and then use the equations of hydrodynamics, together with the corrected constitutive relations

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p) u^\mu u^\nu + p G^{\mu\nu} + f \xi^\mu \xi^\nu + \pi^{\mu\nu} \\ J^\mu &= q u^\mu - f \xi^\mu + j^\mu, \end{aligned} \quad (3.2)$$

(here  $\pi^{\mu\nu}$  and  $j^\mu$  refer to as yet unspecified one and higher derivative corrections to the constitutive relations) to re express this divergence as the sum of a linear form in onshell independent two derivative data and a quadratic form in onshell independent one derivative data. Point wise positivity of the divergence requires the linear form to vanish (this imposes several constraints on the entropy current). Once these conditions are imposed, the divergence of the entropy current is purely a quadratic form in one derivative data. We require this quadratic form to be positive definite. This requirement further constrains the entropy current as well as the first order contributions to  $\pi^{\mu\nu}$  and  $j^\mu$  in a manner we now schematically describe.

As we will see below, the quadratic form so obtained has the property that it vanishes when projected onto a subset of one derivative terms. In other words, all



independent one derivative terms can be divided into  $y$  type ‘entropically dissipative’ terms and  $x$  type entropically nondissipative terms, and the quadratic form takes the schematic form

$$A_{ij}y^iy^j + B_{im}y^ix^m$$

Note that the structure of this quadratic form is preserved under  $x$  redefinitions

$$x_m \rightarrow x_m + C_{mi}y^i$$

but not under analogous redefinitions of  $y^i$ . In other words there exists a well defined subspace of dissipative data but no definite subspace of nondissipative data.

Positivity of the quadratic form described above requires that  $A_{ij}$  is a positive matrix, and  $B_{im} = 0$  for all  $i$  and  $m$ . The last set of constraints yield relations between otherwise apparently independent transport coefficients.<sup>14</sup>

In order to actually implement this process we need first to choose a basis for onshell independent data. As explained in [6] (see e.g. Table 3), at first order in the derivative expansion there exist 7 (4 dissipative and 3 non dissipative) onshell independent scalars, 7 (2 dissipative and 5 nondissipative) onshell independent vectors and 2 (1 dissipative and one nondissipative) independent tensors constructed out of thermodynamical fields and background fields. For the purposes of this section, we will find it useful to choose our onshell independent basis as follows.

*Basis of independent scalars:*

$$\frac{V(\zeta_f)}{(\zeta_f)^2}, \quad (u \cdot \partial H_a), \quad ((\zeta_f) \cdot \partial H_a), \quad a = \{1, 2, 3\}$$

The four of these scalars are dissipative (they vanish in equilibrium) while the remaining three are nondissipative (they are non vanishing in equilibrium, and do not cause entropy production).

*Basis of independent vectors:*

$$\tilde{P}^{\mu\alpha}V_\alpha, \quad \tilde{P}^{\mu\alpha}(\zeta_f)_\beta\sigma_\alpha^\beta, \quad \tilde{P}_\alpha^\mu(\zeta_f)_\nu f^{\nu\alpha}, \quad \tilde{P}_{\alpha\mu}(\zeta_f)_\nu F^{\nu\alpha}, \quad \tilde{P}^{\mu\alpha}\partial_\alpha H_a, \quad a = \{1, 2, 3\}$$

The first two vectors are dissipative (they vanish in equilibrium) and the remaining five vectors are nondissipative.

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<sup>14</sup>Assuming that the matrix  $A$  is positive definite, entropy is always produced whenever any of the  $y^i$  are nonzero. It follows that all  $y^i$  must always vanish in equilibrium. This observation motivates the following definition, utilized in [12]. Expressions that vanish in (arbitrary stationary) equilibrium are referred to as dissipative data. It follows from that entropically dissipative data is necessarily dissipative. However the converse is not necessarily true; it is possible for data to vanish in arbitrary stationary equilibrium but yet be entropically nondissipative. We will see an example of this phenomenon later in this paper.

*Basis of independent tensors*

$$\begin{aligned}\tilde{\sigma}_{\mu\nu} &= \tilde{P}_\mu^\alpha \tilde{P}_\nu^\beta \left[ \frac{\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \tilde{P}^{\lambda\phi} (\nabla_\lambda u_\phi) \eta_{\alpha\beta}}{2} \right], \\ \sigma_{\mu\nu}^{(\zeta_f)} &= \tilde{P}_\mu^\alpha \tilde{P}_\nu^\beta \left[ \frac{\nabla_\alpha (\zeta_f)_\beta + \nabla_\beta (\zeta_f)_\alpha - \tilde{P}^{\lambda\phi} (\nabla_\lambda (\zeta_f)_\phi) \eta_{\alpha\beta}}{2} \right]\end{aligned}$$

The first is dissipative (it vanishes in equilibrium) while the second is nondissipative.

In this paper we wish to constrain the equations of superfluid hydrodynamics presented in a ‘fluid frame’ (see [28] for an explanation of what this means). Throughout this paper we will further restrict our attention to fluid frames with  $\mu_{diss} = 0$  (again see [28] for definitions). This choice still permits the freedom of field redefinitions of the temperature and normal velocity fields (as well as field redefinitions of the superfluid phase, as we will exploit later in this paper). Even though we work specifically frames in which  $\mu_{diss} = 0$  our final results may easily be lifted to an arbitrary  $\mu_{diss} \neq 0$  frame using the frame invariant formalism of [6].

The most general form of the entropy current, consistent with the absence of linear two derivative terms in its divergence was determined in [6] (see equation 3.19 ) and takes the form

$$\begin{aligned}J_S^\mu &= J_{can}^\mu + J_{new}^\mu \\ J_{can}^\mu &= s u^\mu - \nu j^\mu - \frac{u_\nu \pi^{\mu\nu}}{T} \\ J_{new}^\mu &= \sum_a c_a (\partial_\nu H_a) Q^{\mu\nu} + \nabla_\nu (c Q^{\mu\nu}) \\ \text{where } Q^{\mu\nu} &= f(u^\mu (\zeta_f)^\nu - u^\nu (\zeta_f)^\mu)\end{aligned}\tag{3.3}$$

The divergence of  $J_{can}^\mu$  was worked out in [28, 6] (see for example, equation 3.9 [6], and recall we work with  $\mu_{diss} = 0$ ).

$$\nabla_\mu J_{can}^\mu = -\pi^{\mu\nu} \nabla_\mu \left( \frac{u_\nu}{T} \right) + j^\mu V_\mu + (u_\mu j^\mu) (u \cdot \partial \nu)\tag{3.4}$$

The RHS of (3.4) is given schematically by

(one derivative correction to constitutive relation)  $\times$  (entropicallydissipative data),

<sup>15</sup> We will now rewrite the RHS of (3.4) as a quadratic form in the basis of independent dissipative one derivative data chosen above. In order to achieve this we need to rewrite

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<sup>15</sup>Note that the one derivative expressions that appear here are always entropically dissipative, as contributions to changes in the proportional to these one derivative expressions yield quadratic terms in entropy production.

all the  $y$  type terms in (3.4) in terms of the independent basis of dissipative scalars, vectors and tensors listed above. To achieve this we use the equations of motion

$$\begin{aligned} \frac{(\zeta_f) \cdot \partial T}{T} + \mathbf{a} \cdot (\zeta_f) &= RT(V \cdot (\zeta_f)) - \frac{(\zeta_f)^2 K}{\epsilon + P} \\ \frac{(\zeta_f)_\mu (\zeta_f)_\nu \sigma^{\mu\nu}}{(\zeta_f)^2} + \frac{\Theta}{3} &= -T(1 - \mu R) \frac{V \cdot (\zeta_f)}{(\zeta_f)^2} - \frac{\mu K}{\epsilon + P} - \frac{(u \cdot \partial)(\zeta_f)^2}{2(\zeta_f)^2} \end{aligned} \quad (3.5)$$

we find

$$\begin{aligned} \nabla_\mu J_{can}^\mu &= - \left( \frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2} \right) (u \cdot \partial T) + (j \cdot u)(u \cdot \partial \nu) + \frac{(j \cdot (\zeta_f))(V \cdot (\zeta_f))}{(\zeta_f)^2} \\ &\quad + \frac{u_\mu (\zeta_f)_\nu \pi^{\mu\nu}}{T} \left[ RT \left( \frac{V \cdot (\zeta_f)}{(\zeta_f)^2} \right) - \frac{K}{\epsilon + P} \right] - \frac{1}{2T} \left( \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \Theta \\ &\quad - \frac{1}{T} \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \left( \frac{(\zeta_f)_\mu (\zeta_f)_\nu \sigma^{\mu\nu}}{(\zeta_f)^2} + \frac{\Theta}{3} \right) \\ &\quad - 2 \left( \frac{(\zeta_f)_\alpha \pi^{\alpha\nu} \tilde{P}_{\nu\mu} \sigma^{\mu\beta} (\zeta_f)_\beta}{T(\zeta_f)^2} \right) + (j^\mu + Ru_\alpha \pi^{\alpha\mu}) \tilde{P}_{\mu\nu} V^\nu - \frac{1}{T} \tilde{\sigma}_{\mu\nu} \tilde{\pi}^{\mu\nu} \\ &= - \left( \frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2} \right) (u \cdot \partial T) + (j \cdot u)(u \cdot \partial \nu) - \left( \frac{\pi^{\mu\nu} \tilde{P}_{\mu\nu}}{2T} \right) \Theta \\ &\quad + \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \left( \frac{u \cdot \partial (\zeta_f)^2}{2T(\zeta_f)^2} \right) \\ &\quad + \left( \frac{V \cdot (\zeta_f)}{(\zeta_f)^2} \right) \left[ (j \cdot (\zeta_f)) + R(u_\mu (\zeta_f)_\nu \pi^{\mu\nu}) + (1 - \mu R) \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \right] \\ &\quad + \left( \frac{K}{\epsilon + P} \right) \left[ - \left( \frac{-u_\nu (\zeta_f)_\mu \pi^{\mu\nu}}{T} \right) + \nu \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \right] \\ &\quad - 2 \left( \frac{(\zeta_f)_\alpha \pi^{\alpha\nu} \tilde{P}_{\nu\mu} \sigma^{\mu\beta} (\zeta_f)_\beta}{T(\zeta_f)^2} \right) + (j^\mu + Ru_\alpha \pi^{\alpha\mu}) \tilde{P}_{\mu\nu} V^\nu - \frac{1}{T} \tilde{\sigma}_{\mu\nu} \tilde{\pi}^{\mu\nu} \\ &= \sum_{a=1}^3 \mathfrak{S}_a(u \cdot \partial) H_a + \mathfrak{S}_4 \left( \frac{V \cdot (\zeta_f)}{(\zeta_f)^2} \right) - 2\mathfrak{V}_2^\nu \left( \frac{\tilde{P}_{\nu\mu} \sigma^{\mu\beta} (\zeta_f)_\beta}{T(\zeta_f)^2} \right) + \mathfrak{V}_1^\mu \tilde{P}_{\mu\nu} V^\nu - \frac{1}{T} \tilde{\sigma}_{\mu\nu} \tilde{\pi}^{\mu\nu} \end{aligned} \quad (3.6)$$

where

$$\begin{aligned}
\mathfrak{S}_a &= \left[ \left( \frac{s}{\epsilon + P} \right) \frac{\partial}{\partial H_a} \left( \frac{q}{s} \right) \right] \left[ - \left( \frac{-u_\nu (\zeta_f)_\mu \pi^{\mu\nu}}{T} \right) + \nu \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \right] \\
&\quad + \left( \frac{1}{s} \frac{\partial s}{\partial H_a} \right) \left( \frac{\pi^{\mu\nu} \tilde{P}_{\mu\nu}}{2T} \right) - \left( \frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2} \right) \delta_{a,1} + (j \cdot u) \delta_{a,2} \\
&\quad + \left( \frac{1}{2T(\zeta_f)^2} \right) \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \delta_{a,3} \\
\mathfrak{S}_4 &= \left[ (j \cdot (\zeta_f)) + R(u_\mu (\zeta_f)_\nu \pi^{\mu\nu}) + (1 - \mu R) \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \right] \\
\mathfrak{V}_2^\nu &= (\zeta_f)_\alpha \pi^{\alpha\nu} \\
\mathfrak{V}_1^\mu &= (j^\mu + R u_\alpha \pi^{\alpha\mu}) \\
\tilde{\pi}^{\mu\nu} &= \tilde{P}^{\mu\alpha} \tilde{P}^{\nu\beta} \left[ \pi_{\alpha\beta} - \frac{\eta_{\alpha\beta}}{2} \left( \tilde{P}_{\theta\phi} \pi^{\theta\phi} \right) \right] \\
H_1 &= T, \quad H_2 = \nu, \quad H_3 = (\zeta_f)^2
\end{aligned} \tag{3.7}$$

The last line of (3.6) is the final result of this manipulation. It expresses the divergence of the entropy current as a linear sum over the four dissipative onshell scalars and two dissipative onshell vectors and one dissipative tensor listed earlier in this subsection. These expressions appear multiplied by frame invariant linear combinations of  $\pi^{\mu\nu}$  and  $j^\mu$ .

The frame invariant quantities  $\mathfrak{S}_a$  and  $\mathfrak{V}_a$  will be used extensively below. For later use we will find it useful to regard these quantities as linear functions of  $\pi^{\mu\nu}$  and  $j^\mu$ , i.e.

$$\mathfrak{S}_a = \mathfrak{S}_a(\pi^{\mu\nu}, j^\mu), \quad \mathfrak{V}_a = \mathfrak{V}_a(\pi^{\mu\nu}, j^\mu) \tag{3.8}$$

The divergence of the ‘new’ part of the entropy current,  $J_{new}^\mu$  (see (3.3)) is given by

$$\begin{aligned}
&\nabla_\mu J_{new}^\mu \\
&= \sum_{(a,b)} f \left( \frac{\partial c_a}{\partial H_b} - \frac{\partial c_b}{\partial H_a} \right) ((\zeta_f) \cdot \partial H_a) (u \cdot \partial H_b) + \sum_a (\partial_\nu H_a) \nabla_\mu Q^{\mu\nu} \\
&= \sum_{(a,b)} f \left( \frac{\partial c_a}{\partial H_b} - \frac{\partial c_b}{\partial H_a} \right) ((\zeta_f) \cdot \partial H_a) (u \cdot \partial H_b) + \sum_a (\partial_\mu H_a) \tilde{P}_\nu^\mu (\nabla_\alpha Q^{\alpha\nu}) \\
&\quad - \sum_a \left[ (u \cdot \partial H_a) (u_\nu \nabla_\mu Q^{\mu\nu}) + ((\zeta_f) \cdot \partial H_a) \left( \frac{(\zeta_f)_\nu \nabla_\mu Q^{\mu\nu}}{(\zeta_f)^2} \right) \right]
\end{aligned} \tag{3.9}$$

where  $Q^{\mu\nu}$  was defined in (3.3).

Using equations of motion we can express  $(u_\nu \nabla_\mu Q^{\mu\nu})$ ,  $\left( \frac{(\zeta_f)_\nu \nabla_\mu Q^{\mu\nu}}{(\zeta_f)^2} \right)$  and  $\tilde{P}_\nu^\mu (\nabla_\alpha Q^{\alpha\nu})$  in terms of the onshell independent basis scalars of this subsection (spanned by  $(u \cdot \partial H_a)$ ,  $((\zeta_f) \cdot \partial H_a)$ ,  $\left( \frac{V \cdot (\zeta_f)}{(\zeta_f)^2} \right)$ ) and vectors (spanned by  $\tilde{P}^{\mu\alpha} V_\alpha$ ,  $\tilde{P}^{\mu\alpha} (\zeta_f)_\beta \sigma_\alpha^\beta$ ,  $\tilde{P}^{\mu\alpha} \partial_\alpha H_a$ ).

$$\begin{aligned}
(u_\nu \nabla_\mu Q^{\mu\nu}) &= s(u.\partial) \left( \frac{f\mu}{s} \right) + \left( 1 - \frac{f(\zeta_f)^2}{\epsilon + P} \right) K + f \left( \frac{(\zeta_f).\partial T}{T} \right) - f(V.(\zeta_f)) \\
\left( \frac{(\zeta_f)_\nu \nabla_\mu Q^{\mu\nu}}{(\zeta_f)^2} \right) &= s(u.\partial) \left( \frac{f}{s} \right) + fT \left[ (1 - \mu R) \left( \frac{V.(\zeta_f)}{(\zeta_f)^2} \right) + \frac{\nu K}{\epsilon + P} + \frac{u.\partial(\zeta_f)^2}{T(\zeta_f)^2} \right] \\
\tilde{P}_\nu^\mu (\nabla_\alpha Q^{\alpha\nu}) &= -P_\nu^\mu \sum_a f c_a [T(1 - \mu R) V^\nu + 2(\zeta_f)_\alpha \sigma^{\nu\alpha}]
\end{aligned} \tag{3.10}$$

From equations (3.6), (3.7), (3.9) and (3.10) we conclude that, no matter what form the fluid constitutive relations take, the divergence of the entropy current cannot contain any expressions of the form  $((\zeta_f).\partial H_a)^2$  or  $(\tilde{P}^{\mu\nu} \partial_\mu H_a \partial_\nu H_b)$ . In other words the scalars  $(\zeta_f).\partial H_a$  and the vectors  $(\tilde{P}^{\mu\nu} \partial_\mu H_a)$  are nondissipative. It follows that the positivity of  $(\nabla_\mu J_S^\mu)$  requires that the divergence contain no term linear in  $((\zeta_f).\partial H_a)$  or  $(\tilde{P}^{\mu\nu} \partial_\mu H_a)$  (see e.g. [6] for repeated use of similar arguments.) To ensure this  $\pi^{\mu\nu}$  and  $j^\mu$  have to satisfy the following conditions.

$$\begin{aligned}
\mathfrak{S}_a &= - \sum_{b=1}^3 ((\zeta_f).\partial H_b) \left\{ f \left( \frac{\partial c_b}{\partial H_a} - \frac{\partial c_a}{\partial H_b} \right) - \frac{f c_a}{T} \delta_{b,1} + \frac{f c_b}{(\zeta_f)^2} \delta_{a,3} \right. \\
&\quad \left. + c_b \left[ s \frac{\partial}{\partial H_a} \left( \frac{f}{s} \right) + \left( \frac{s\nu}{\epsilon + P} \right) \frac{\partial}{\partial H_a} \left( \frac{q}{s} \right) \right] \right\} \\
&\quad + \text{dissipative terms} \\
&= - \sum_{b=1}^3 ((\zeta_f).\partial H_b) \left\{ \left[ \frac{\partial}{\partial H_a} (f c_b) - \frac{f}{T} \frac{\partial}{\partial H_b} (T c_a) \right] + \frac{f c_b}{(\zeta_f)^2} \delta_{a,3} \right. \\
&\quad \left. + c_b \left[ -\frac{1}{s} \frac{\partial s}{\partial H_a} + \left( \frac{s\nu}{\epsilon + P} \right) \frac{\partial}{\partial H_a} \left( \frac{q}{s} \right) \right] \right\} \\
&\quad + \sum_{b=1}^3 M_{ab} (u.\partial H_b) + M_{a4} \left( \frac{V.(\zeta_f)}{(\zeta_f)^2} \right) \quad (a = \{1, 2, 3\}) \\
\mathfrak{S}_4 &= (j.(\zeta_f)) + R(u_\mu (\zeta_f)_\nu \pi^{\mu\nu}) + (1 - \mu R) \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) + \text{dissipative terms} \\
&= - \sum_b ((\zeta_f).\partial H_b) f T (1 - \mu R) c_b + \sum_{b=1}^3 M_{4b} (u.\partial H_b) + M_{44} \left( \frac{V.(\zeta_f)}{(\zeta_f)^2} \right)
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\mathfrak{V}_{1\mu} &= (j^\nu + Ru_\alpha \pi^{\alpha\nu}) \tilde{P}_{\mu\nu} = T(1 - \mu R) f \sum_b c_b (\tilde{P}_\mu^\nu \partial_\nu H_b) + \text{dissipative terms} \\
&= T(1 - \mu R) f \sum_b c_b (\tilde{P}_\mu^\nu \partial_\nu H_b) + N_{11} (\tilde{P}_{\mu\nu} V^\nu) - N_{12} \left( \frac{\tilde{P}_{\mu\beta} (\zeta_f)_\alpha \sigma^{\alpha\beta}}{2T(\zeta_f)^2} \right) \\
\mathfrak{V}_{2\mu} &= (\zeta_f)_\alpha \pi^{\alpha\nu} \tilde{P}_{\nu\mu} = -T(\zeta_f)^2 f \sum_b c_b (\tilde{P}_\mu^\nu \partial_\nu H_b) + \text{dissipative terms} \\
&= -T(\zeta_f)^2 f \sum_b c_b (\tilde{P}_\mu^\nu \partial_\nu H_b) + N_{21} (\tilde{P}_{\mu\nu} V^\nu) - N_{22} \left( \frac{\tilde{P}_{\mu\beta} (\zeta_f)_\alpha \sigma^{\alpha\beta}}{2T(\zeta_f)^2} \right) \\
\tilde{\pi}^{\mu\nu} &= \tilde{P}^{\mu\alpha} \tilde{P}^{\nu\beta} \left[ \pi_{\alpha\beta} - \frac{\eta_{\alpha\beta}}{2} (\tilde{P}_{\theta\phi} \pi^{\theta\phi}) \right] = \text{dissipative term} = -\eta \tilde{\sigma}^{\mu\nu}
\end{aligned} \tag{3.12}$$

where  $M$  is a  $4 \times 4$  matrix of dissipative transport coefficients in the scalar sector and  $N$  is a  $2 \times 2$  matrix of dissipative transport coefficients in the vector sector.

Equations (3.11),(3.12) are the main result of this subsection. It expresses the equality type constraints that follow from the local second law. Once (3.11),(3.12) are satisfied the final expressions for the divergence of the entropy current takes the following form.

$$\begin{aligned}
&\nabla_\mu J_s^\mu \\
&= - \sum_{a,b} c_a \left\{ \mu s \frac{\partial (\frac{f}{s})}{\partial H_b} + s \left( 1 - \frac{f(\zeta_f)^2}{\epsilon + p} \right) \frac{\partial (\frac{g}{s})}{\partial H_b} + f (T\delta_{b,2} + \nu\delta_{b,1}) \right\} (u.\partial H_a)(u.\partial H_b) \\
&\quad - f(V.(\zeta_f)) \sum_a c_a (u.\partial H_a) \\
&\quad + \sum_{a,b=1}^3 M_{ab} (u.\partial H_a)(u.\partial H_b) + \sum_{a=1}^3 (M_{4a} + M_{a4}) (u.\partial H_a) \left( \frac{V.(\zeta_f)}{(\zeta_f)^2} \right) + M_{44} \left( \frac{V.(\zeta_f)}{(\zeta_f)^2} \right)^2 \\
&\quad + \tilde{P}_{\mu\nu} [N_{11} (V^\mu V^\nu) + (N_{12} + N_{21}) (V^\mu (\zeta_f)_\alpha \sigma^{\alpha\nu}) + N_{22} ((\zeta_f)_\alpha (\zeta_f)_\beta \sigma^{\alpha\mu} \sigma^{\beta\nu})] \\
&\quad + \frac{\eta}{T} \tilde{\sigma}_{\mu\nu} \tilde{\sigma}^{\mu\nu}
\end{aligned} \tag{3.13}$$

The positivity of this quadratic form imposes additional inequality type constraints on transport coefficients that we will not further explore here.

### 3.2 Constraints from the partition function

In this subsection we now reproduce the conditions (3.11),(3.12) using considerations independent of those of the previous subsection. The procedure we adopt is very similar to that described in [12], and we describe it only briefly, highlighting only those elements of the analysis that are unique to the superfluid.

The starting point of our analysis is the expressions (2.22) and (2.23) which represent the first order for the corrections to the stress tensor and charge current that

follow by varying the local action for the Goldstone mode w.r.t. the metric and background gauge field. Once we substitute in the solution for the field  $\xi^\mu(x)$ , according to its equations of motion, (2.22) and (2.23) yield first order corrections  $\delta T_{\mu\nu}$  and  $\delta J^\mu$  to the values of the stress tensor and charge current in thermal equilibrium.

From the hydrodynamical point of view,  $\delta T_{\mu\nu}$  and  $\delta J^\mu$  are the first order contributions in (3.2) once we substitute

$$T(x) = \hat{T}(x) + T_1(x), \quad \mu(x) = \hat{\mu}(x) + \mu_1(x), \quad u^\mu(x) = \hat{u}^\mu + u_1^\mu(x) \quad (3.14)$$

into those expressions. Here  $T_1(x)$ ,  $\mu_1(x)$  and  $u_1^\mu(x)$  are the first derivative corrections to the equilibrium configurations of temperature, chemical potential and velocity.

Upon substituting (3.14) into (3.2) we get first derivative contributions of two sorts. First we have the corrections to constitutive relations evaluated on the zero order equilibrium configurations  $\Pi^{\mu\nu}(\hat{T}, \hat{\mu}, \hat{u}^\mu, (\zeta^{eq})^\mu)$  and  $j^\mu(\hat{T}, \hat{\mu}, \hat{u}^\mu, (\zeta^{eq})^\mu)$ . Second we have contributions from terms proportional to  $T_1$ ,  $\mu_1$  and  $u_1^\mu$  when (3.14) is plugged into the perfect fluid constitutive relations. Contributions of the second sort, however, *precisely* cancel out in the frame invariant linear combinations  $\mathfrak{S}_a$  ( $a = 1 \dots 4$ ) and  $\mathfrak{V}_a$  ( $a = 1 \dots 2$ ). In other words

$$\begin{aligned} \mathfrak{S}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{S}_a(\pi_{\mu\nu}, j_\mu) \\ \mathfrak{V}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{V}_a(\pi_{\mu\nu}, j_\mu) \end{aligned} \quad (3.15)$$

( $\pi^{\mu\nu}$  and  $j^\mu$  on the RHS of (3.15) are evaluated on the zero order equilibrium configurations). In the general formulation of hydrodynamics, however, it is precisely the frame invariants that appear on the RHS of (3.15) that are expanded in the most general symmetry allowed constitutive relations (see e.g. [6] )

$$\begin{aligned} \mathfrak{S}_a(\pi^{\mu\nu}, j^\mu) &= \alpha_{am} S^m \quad (a = 1 \dots 4), \quad (m = 1 \dots 7) \\ \mathfrak{V}_a^\mu(\pi^{\mu\nu}, j^\mu) &= \gamma_{am} V_m^\mu \quad (a = 1 \dots 2), \quad (m = 1 \dots 5) \end{aligned} \quad (3.16)$$

where  $S^m$  and  $V_m^\mu$  are the independent one derivative scalars and vectors and the coefficients  $\alpha_{am}$  and  $\gamma_{am}$  are arbitrary functions of the scalars  $T$ ,  $\mu$  and  $\xi^\mu \xi_\mu$ .

$\alpha_{am}$  and  $\gamma_{am}$  are the constitutive coefficients we wish to constrain, and this is achieved as follows. In the LHS of (3.15) we substitute the expressions (2.22) and (2.23) for  $\delta T^{\mu\nu}$  and  $\delta J^\mu$ . This determines the LHS of (3.15) completely in terms of the functions  $f_1$ ,  $f_2$  and  $f_3$  that appear in the partition function. In the RHS of (3.15) we substitute (3.16), and evaluate these expressions in equilibrium

$$T = \hat{T}, \quad \mu = \hat{\mu}, \quad \zeta = \zeta^{eq}.$$

Under the last substitution those of  $S^m$  and  $V^m$  that are dissipative vanish. The non dissipative one derivative scalars and vectors evaluate to geometric expressions.

Equating the coefficients of these expressions we determine  $\alpha_{am}$  and  $\gamma_{am}$  for those values of  $m$  that correspond to non dissipative terms. In other words this procedure completely determines all non dissipative transport coefficients.<sup>16</sup>

In the rest of this subsection we implement the procedure described above to explicitly determine all nondissipative transport coefficients in terms of the three free functions  $f_1$ ,  $f_2$  and  $f_3$  that enter the local action for the Goldstone field. We demonstrate that our results agree exactly with (3.11),(3.12), obtained from the local form of the second law, once we identify the three unknown functions  $c_1$ ,  $c_2$  and  $c_3$  in the entropy current of the previous subsection in terms of the functions in the partition function according to

$$c_1 = \frac{f_1}{fT} + \frac{1}{T} \frac{\partial f_3}{\partial T}, \quad c_2 = \frac{f_2}{fT} + \frac{1}{T} \frac{\partial f_3}{\partial \nu}, \quad c_3 = \frac{1}{T} \frac{\partial f_3}{\partial \zeta^2} \quad (3.17)$$

We will also demonstrate that the identification (3.17) may be argued for directly by comparing the thermodynamical entropy in equilibrium with the integral of the equilibrium entropy current over a spatial slice.

It will be useful in the computation below to note that  $P^{\mu\nu}$  and  $\tilde{P}^{\mu\nu}$  are given by

$$P_{ij} = g_{ij}, \quad \tilde{P}_{ij} = g_{ij} - \frac{\zeta_i^{eq} \zeta_j^{eq}}{(\zeta^{eq})^2}$$

We turn now to the explicit computation, starting with the vectors.

$$\begin{aligned} \mathfrak{V}_{10}(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{V}_{20}(\delta T_{\mu\nu}, \delta J_\mu) = 0 \\ \mathfrak{V}_{1i}(\delta T_{\mu\nu}, \delta J_\mu) &= \tilde{P}_{ij} \left( \delta J^j + \hat{R} \hat{u}^0 \delta T_0^j \right) \\ &= \tilde{P}_{ij} \left( \delta J^j - Re^{-\sigma} A_0 \delta J^j \right) \\ &= (1 - \hat{\mu} \hat{R}) \tilde{P}_{ij} \delta J^j \\ &= \tilde{P}_{ij} g^{jk} (1 - \hat{\mu} \hat{R}) \left( \frac{f_1}{\hat{T}} \partial_k \hat{T} + \frac{f_2}{\hat{T}} \partial_k \hat{\nu} + \frac{f}{\hat{T}} \partial_k f_3 \right) \\ \mathfrak{V}_{2i}(\delta T_{\mu\nu}, \delta J_\mu) &= \tilde{P}_{ij} \zeta_k \delta T^{kj} = -\zeta^2 \tilde{P}_{ij} \delta J^j \\ &= -\tilde{P}_{ij} g^{jk} \left( \frac{f_1}{\hat{T}} \partial_k \hat{T} + \frac{f_2}{\hat{T}} \partial_k \hat{\nu} + \frac{f}{\hat{T}} \partial_k f_3 \right) \end{aligned} \quad (3.18)$$

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<sup>16</sup>There is an important subtlety here. All of the operations described above may only be performed in equilibrium, i.e. once we have solved for  $(\zeta^{eq})^\mu$  as a function of background fields and substituted this back into the partition function. We implement our programme without explicitly solving, simply by treating  $(\zeta^{eq})^\mu(x)$  as formally independent of the other background fields, except for those local combinations of  $(\zeta^{eq})^\mu$  that appear in terms of its equation of motion and derivatives thereof. The reason for this is that the expressions for  $\xi^\mu$  as a function of background fields is highly nonlocal. The only situation in which cancellations are possible between local expressions in  $(\zeta^{eq})^\mu$  and local expressions in the background fields is when we get derivatives combining with  $(\zeta^{eq})^\mu$  in the form of the  $\phi$  equations of motion.



The last line of (3.18) exactly matches (3.11),(3.12)) upon using the identification of the parameters (3.17).

We turn next to the scalars; let us start with  $\mathfrak{S}_4$ .

$$\begin{aligned}\mathfrak{S}_4(\delta T_{\mu\nu}, \delta J_\mu) &= (j \cdot \zeta) + R(u_\mu \zeta_\nu \pi^{\mu\nu}) + (1 - \mu R) \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \\ &= - (1 - \hat{\mu} \hat{R}) \left[ f_1(\zeta^{eq} \cdot \partial \hat{T}) + f_2(\zeta^{eq} \cdot \partial \hat{\nu}) + f(\zeta^{eq} \cdot \partial f_3) \right] \\ &= \hat{T} (1 - \hat{\mu} \hat{R}) \sum_b f c_b(\zeta^{eq} \cdot \partial H_b)\end{aligned}\quad (3.19)$$

In the last step we have used (3.17), and have obtained manifest agreement with (3.11),(3.12).

Next we shall calculate the remaining three scalars  $\mathfrak{S}_a$ ,  $a = \{1, 2, 3\}$ . The algebraic manipulations here are a little more involved than in previous cases, and we provide some details.

$$\begin{aligned}\mathfrak{S}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \left[ \left( \frac{s}{\epsilon + P} \right) \frac{\partial}{\partial H_a} \left( \frac{q}{s} \right) \right] \left[ - \left( \frac{-u_\nu \zeta_\mu \pi^{\mu\nu}}{T} \right) + \nu \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \right] \\ &\quad + \left( \frac{1}{s} \frac{\partial s}{\partial H_a} \right) \left( \frac{\pi^{\mu\nu} \tilde{P}_{\mu\nu}}{2T} \right) - \left( \frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2} \right) \delta_{a,1} + (j \cdot u) \delta_{a,2} \\ &\quad + \left( \frac{1}{2T\zeta^2} \right) \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \delta_{a,3}, \quad (a = \{1, 2, 3\})\end{aligned}\quad (3.20)$$

The first line in (3.20) can be evaluated as

$$\begin{aligned}&\left[ \left( \frac{s}{\epsilon + P} \right) \frac{\partial}{\partial H_a} \left( \frac{q}{s} \right) \right] \left[ - \left( \frac{-u_\nu \zeta_\mu \pi^{\mu\nu}}{T} \right) + \nu \left( \pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu} \right) \right] \\ &= \left[ \left( \frac{s}{\epsilon + P} \right) \frac{\partial}{\partial H_a} \left( \frac{q}{s} \right) \right] \left[ -\hat{\nu} f(\zeta^{eq} \cdot \partial f_3) - \hat{\nu} f_2(\zeta^{eq} \cdot \partial \hat{\nu}) - \hat{\nu} f_1(\zeta^{eq} \cdot \partial \hat{T}) \right] \\ &= - \left[ \left( \frac{s\hat{\nu}}{\epsilon + P} \right) \frac{\partial}{\partial H_a} \left( \frac{q}{s} \right) \right] \sum_b c_b(\zeta^{eq} \cdot \partial H_b)\end{aligned}\quad (3.21)$$

In the last step we have used (3.17).

The second line of (3.20) may be evaluated as follows

$$\begin{aligned}&\left( \frac{1}{s} \frac{\partial s}{\partial H_a} \right) \left( \frac{\pi^{\mu\nu} \tilde{P}_{\mu\nu}}{2T} \right) \\ &= \left( \frac{1}{s} \frac{\partial s}{\partial H_a} \right) \left[ \frac{f_1}{\hat{T}}(\zeta^{eq} \cdot \partial) \hat{T} + \frac{f_2}{\hat{T}}(\zeta^{eq} \cdot \partial) \hat{\nu} + \frac{f}{\hat{T}}(\zeta^{eq} \cdot \partial) f_3 \right] \\ &= \left( \frac{1}{s} \frac{\partial s}{\partial H_a} \right) \sum_b f c_b(\zeta^{eq} \cdot \partial H_b)\end{aligned}\quad (3.22)$$

In the last step we have used (3.17).

Finally we evaluate the last three terms of (3.20) together.

$$\begin{aligned}
& - \left( \frac{u_\mu u_\nu \pi^{\mu\nu}}{T^2} \right) \delta_{a,1} + (j \cdot u) \delta_{a,2} + \left( \frac{\pi^{\mu\nu} P_{\mu\nu} - \frac{3}{2} \pi^{\mu\nu} \tilde{P}_{\mu\nu}}{2T\zeta^2} \right) \delta_{a,3} \\
= & - \left[ \frac{\partial}{\partial \hat{T}} \left( \frac{f_1}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \hat{T}) + \frac{\partial}{\partial \hat{T}} \left( \frac{f_2}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \nu) + \frac{\partial}{\partial \hat{T}} \left( \frac{f}{\hat{T}} \right) (\zeta^{eq} \cdot \partial f_3) - \frac{f}{\hat{T}} \zeta^{eq} \cdot \partial \left( \frac{f_1}{f} \right) \right] \delta_{a,1} \\
& - \left[ \frac{\partial}{\partial \nu} \left( \frac{f_1}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \hat{T}) + \frac{\partial}{\partial \nu} \left( \frac{f_2}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \nu) + \frac{\partial}{\partial \nu} \left( \frac{f}{\hat{T}} \right) (\zeta^{eq} \cdot \partial f_3) - \frac{f}{\hat{T}} \zeta^{eq} \cdot \partial \left( \frac{f_2}{f} \right) \right] \delta_{a,2} \\
& - \frac{1}{\hat{T}} \left[ \frac{\partial}{\partial (\zeta^{eq})^2} \left( \frac{f_1}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \hat{T}) + \frac{\partial}{\partial (\zeta^{eq})^2} \left( \frac{f_2}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \nu) + \frac{\partial}{\partial (\zeta^{eq})^2} \left( \frac{f}{\hat{T}} \right) (\zeta^{eq} \cdot \partial f_3) \right] \delta_{a,3} \\
& - \left[ \frac{f_1}{\hat{T}} (\zeta^{eq} \cdot \partial \hat{T}) + \frac{f_2}{\hat{T}} (\zeta^{eq} \cdot \partial \nu) + \frac{f}{\hat{T}} (\zeta^{eq} \cdot \partial f_3) \right] \delta_{a,3} \\
= & \sum_b (\zeta^{eq} \cdot \partial H_b) \left[ - \left( \frac{\partial f_3}{\partial H_b} \right) \frac{\partial}{\partial H_a} \left( \frac{f}{\hat{T}} \right) + \frac{f}{\hat{T}} \frac{\partial}{\partial H_b} \left( \frac{f_1}{f} \right) \delta_{a,1} + \frac{f}{\hat{T}} \frac{\partial}{\partial H_b} \left( \frac{f_2}{f} \right) \delta_{a,2} \right] \\
& - \frac{\partial}{\partial H_a} \left( \frac{f_1}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \hat{T}) - \frac{\partial}{\partial H_a} \left( \frac{f_2}{\hat{T}} \right) (\zeta^{eq} \cdot \partial \nu) \\
& - \left[ \frac{f_1}{\hat{T}} (\zeta^{eq} \cdot \partial \hat{T}) + \frac{f_2}{\hat{T}} (\zeta^{eq} \cdot \partial \nu) + \frac{f}{\hat{T}} (\zeta^{eq} \cdot \partial f_3) \right] \delta_{a,3} \\
= & - \sum_b (\zeta^{eq} \cdot \partial H_b) \left[ \frac{\partial}{\partial H_a} (f c_b) - \frac{f}{\hat{T}} \frac{\partial}{\partial H_b} (\hat{T} c_a) + \frac{f c_b}{(\zeta^{eq})^2} \delta_{a,3} \right]
\end{aligned} \tag{3.23}$$

In the last step we have used (3.17).

Combining (3.21), (3.22) and (3.23) it is straightforward to verify that the expressions for  $\mathfrak{S}_a$ ,  $a = \{1, 2, 3, 4\}$  as derived from partition function in this subsection, match exactly with (3.11), (3.12). Note that both methods leave dissipative contributions to constitutive relations completely unconstrained.

### 3.3 Entropy from the partition function

In this subsection we will explain how the nondissipative part of the entropy current of the superfluid may be read off in a rather direct way from the partition function. Our analysis is largely structural, and applies equally well to normal (non super) fluids. However our presentation applies only at first order in the derivative expansion.

For any system the entropy  $S_T$  in equilibrium may be evaluated from the logarithm of partition function  $W = \ln Z$  via the thermodynamical relation

$$S_T = W + T_0 \frac{\partial W}{\partial T_0} \tag{3.24}$$

We will now rewrite this expression in terms of the goldstone action that generates the partition function. Let this action take the form

$$S = \int \sqrt{g} \mathcal{L} d^3x$$

and also suppose

$$\mathcal{L}^{eq} = \mathcal{L}(\zeta_\mu = \zeta_\mu^{eq})$$

Now we can think of the partition function as

$$W = S(\hat{T}, \hat{\mu}, T_0 a_i, \zeta_\mu^{eq}) = \int \sqrt{g} \mathcal{L}^{eq}(\hat{T}, \hat{\mu}, T_0 a_i, \zeta_\mu^{eq}) d^3x$$

Using the simple rescaling of the time coordinate employed in subsection 2.3.1 of [12] one may show that

$$\begin{aligned} T_0 \frac{\partial \hat{T}}{\partial T_0} &= \hat{T} \\ T_0 \frac{\partial \hat{\nu}}{\partial T_0} &= -\hat{\nu} \\ T_0 \frac{\partial a_i}{\partial T_0} &= 0 \\ T_0 \frac{\partial \zeta_i}{\partial T_0} &= 0 \end{aligned} \tag{3.25}$$

It follows that

$$\begin{aligned} & \frac{\partial W}{\partial T_0} \\ &= \int d^3y \sqrt{g} \left[ \left( \frac{\delta \mathcal{L}^{eq}}{\delta \hat{T}(y)} \right) \left( \frac{\partial \hat{T}(y)}{\partial T_0} \right) + \left( \frac{\delta \mathcal{L}^{eq}}{\delta \hat{\nu}(y)} \right) \left( \frac{\partial \hat{\nu}(y)}{\partial T_0} \right) + \left( \frac{\delta \mathcal{L}^{eq}}{\delta a_i(y)} \right) \left( \frac{\partial a_i(y)}{\partial T_0} \right) \right] \\ &= \int d^3y \sqrt{g} e^{-\sigma} \left[ \frac{T_{00}}{T_0^2} + \frac{\hat{\nu} J_0}{T_0} + a_i \left( \frac{T_0^i + A_0 \delta J^i}{\hat{T}^2} \right) \right] \\ &= \int d^3y \sqrt{g} e^{\sigma} \left[ \frac{1}{T_0^2} (T_{00} e^{-2\sigma} + a_i T_0^i) + \frac{\hat{\nu}}{T_0} (J_0 e^{-2\sigma} + a_i J^i) \right] \\ &= \int d^3y \sqrt{-G} \left[ \frac{1}{T_0^2} \left( -\frac{T_{00}}{G_{00}} + \frac{G_{0i}}{G_{00}} T_0^i \right) + \frac{\hat{\nu}}{T_0} \left( -\frac{J_0}{G_{00}} + \frac{G_{0i}}{G_{00}} J^i \right) \right] \\ &= \int d^3y \frac{\sqrt{-G}}{T_0} \left[ -\frac{T_0^0}{T_0} - \hat{\nu} J^0 \right] \\ &= \int d^3y \frac{\sqrt{-G}}{T_0} \left[ -\frac{\hat{u}^\mu T_\mu^0}{\hat{T}} - \hat{\nu} J^0 \right] \end{aligned} \tag{3.26}$$

so that

$$S_T = W + \frac{\partial W}{\partial T_0} = \int d^3y \frac{\sqrt{-G}}{T_0} \left[ \hat{T} \mathcal{L}^{eq} - \frac{\hat{u}^\mu T_\mu^0}{\hat{T}} - \hat{\nu} J^0 \right] \tag{3.27}$$

This expression may be expanded to first order in derivatives employing

$$\begin{aligned}\mathcal{L}^{eq} &= \frac{\hat{P}}{\hat{T}} + \mathcal{L}_1^{eq} \\ T_0^0 &= (T_0^0)_{perf} + \delta T_0^0 \\ J^0 &= J_{perf}^0 + \delta J^0\end{aligned}\tag{3.28}$$

where, from (2.14)

$$\begin{aligned}(T_0^0)_{perf} &= -\hat{\epsilon} - \hat{f}e^{-2\sigma}A_0^2 - \hat{f}A_0a^i\zeta_i^{eq} \\ J_{perf}^0 &= e^{-\sigma}\hat{q} - \hat{f}(e^{-2\sigma}A_0 + a^i\zeta_i^{eq}),\end{aligned}\tag{3.29}$$

$\delta T_0^0$  is defined in (2.23) and  $\delta J^0$  is defined in (2.22).

Using the Gibbs Duham relation

$$s = \frac{P + \epsilon - q\mu}{T}$$

we find that

$$\begin{aligned}S_T &= \int d^3y \sqrt{g} \hat{s} \\ &+ \int d^3y \frac{\sqrt{-G}}{T_0} \left[ \hat{T} \mathcal{L}_1^{eq} - \frac{\hat{u}^\mu \delta T_\mu^0}{\hat{T}} - \hat{\nu} \delta J^0 \right]\end{aligned}\tag{3.30}$$

(all proportional to  $\hat{f}$  cancel out at zero order in the derivative expansion).

Now let us recall that

$$\delta T_\nu^\mu = (\pi_\nu^\mu)_0 + (T_\nu^\mu)_1^{perf}$$

where  $(\pi_\nu^\mu)_0$  refers to  $\pi_\nu^\mu$  evaluated on the zero order equilibrium solution and  $(T_\nu^\mu)_1^{perf}$  refers to the one derivative correction in  $T_\nu^\mu$  from the first order correction to the equilibrium solution. Similarly

$$\delta J^\mu = (j^\mu)_0 + (J_{perf}^\mu)^1.$$

It follows that

$$\begin{aligned}S_T &= \int d^3y \sqrt{-G} \left[ \hat{s} \hat{u}^0 - \frac{\hat{u}^\mu \delta (T_\mu^0)_1^{perf}}{\hat{T}} - \hat{\nu} \delta (J^0)_1^{perf} \right] \\ &+ \int d^3y \frac{\sqrt{-G}}{T_0} \left[ \hat{T} \mathcal{L}_1^{eq} - \frac{\hat{u}^\mu \delta \pi_\mu^0}{\hat{T}} - \hat{\nu} \delta j^0 \right]\end{aligned}\tag{3.31}$$

so that

$$\begin{aligned}S_T &= \int d^3y \sqrt{-G} s u^0 \\ &+ \int d^3y \frac{\sqrt{-G}}{T_0} \left[ \hat{T} \mathcal{L}_1^{eq} - \frac{\hat{u}^\mu \delta \pi_\mu^0}{\hat{T}} - \hat{\nu} \delta j^0 \right]\end{aligned}\tag{3.32}$$

where  $su^0$  in (3.32) refers to the entropy evaluated on the first order corrected solution. In going from (3.31) to (3.32) we have used the fact that the frame invariance (see [6] for a definition and extensive discussion of frame invariance) of the canonical entropy current

$$J_{can}^\mu = su^\mu - \nu j^\mu - \frac{u_\nu \pi^{\mu\nu}}{T}$$

implies that

$$su^\mu - \hat{s}\hat{u}^\mu + \nu(J^\nu)^1_{perf} + \frac{u_\nu(T^{\mu\nu})^1_{perf}}{T} = 0.$$

It follows from (3.32) that

$$S_T = \int d^3y \sqrt{-G} \left[ J_{can}^0 + \frac{1}{T_0} \hat{T} \mathcal{L}_1^{eq} \right] \quad (3.33)$$

Comparing with

$$J_S^\mu = J_{can}^\mu + J_{new}^\mu \quad (3.34)$$

we conclude that

$$\int d^3y \sqrt{-G} J_{new}^0 = \int d^3y \sqrt{-G} \frac{\hat{T}}{T_0} \mathcal{L}_1^{eq} \quad (3.35)$$

In other words the integral of  $J_{new}^0$  matches with the first order correction to the Goldstone action. (3.35) is the principal formal result of this subsection. It expresses a very simple relationship between the correction to the canonical entropy current of our system and the first order correction to the partition function.

To what extent does (3.35) determine  $J_{new}^\mu$ ? The most general first order correction to the entropy current takes the form

$$J_{new}^\mu = S_u u^\mu + S_\zeta \zeta^\mu + V_s^\mu \quad (3.36)$$

where  $S_u$  and  $S_\zeta$  are first order scalars while  $V_s^\mu$  is a first order vector. Notice that, to first order,  $X^\mu = S_\zeta \zeta^\mu + V_s^\mu$  is orthogonal to  $\hat{u}$ . It follows immediately from this observation that

$$X^0 = -a_i X^i$$

Plugging this relation into (3.35) we conclude that the contribution from  $X^\mu$  to the total entropy is not Kaluza Klein gauge invariant and so must vanish (see [12] for a discussion on related issues). It follows that  $S_\zeta$  and  $V_s^\mu$  vanish in equilibrium. Upto dissipative corrections, therefore, it follows that

$$J_{new}^\mu = S_u u^\mu \quad (3.37)$$

Now comparing with (3.35) it follows that

$$\int d^3y \sqrt{g} (S_u - \mathcal{L}_1^{eq}) = 0$$

so that

$$S_u = \mathcal{L}_1^{eq} + \text{total derivatives} \quad (3.38)$$

Let us now turn to the case at hand.  $\mathcal{L}_1^{eq}$  was listed in (2.17). It is easily verified that there exist no total derivative scalars at one derivative order. Consequently we conclude that

$$S_u = \frac{f_1}{\hat{T}}(\zeta \cdot \partial)\hat{T} + \frac{f_2}{\hat{\nu}}(\zeta \cdot \partial)\hat{\nu} - f_3 \nabla_i \left( \frac{f}{\hat{T}} \zeta^i \right) + \text{dissipative}$$

It is not difficult to verify that this expression, together with (3.37), agree exactly with (3.3) in equilibrium once we employ the identification of parameters (3.17).

In summary, the positive divergence entropy current - which we determined earlier in this paper - is also uniquely determined by comparison with the partition function for parity even superfluids at first order in the derivative expansion.

### 3.4 Consistency with field redefinitions

We will now verify that the dependence of the constitutive relations and entropy current of the superfluid on  $f_3$  is consistent with the transformation (2.21) of  $f_3$  under the field redefinition (2.18).

Recall that the stress tensor and currents of our system take the form

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + P)u^\mu u^\nu + P G^{\mu\nu} + f \xi^\mu \xi^\nu + \pi^{\mu\nu} \\ J^\mu &= q u^\mu - f \xi^\mu + j^\mu \\ J_s^\mu &= J_{can}^\mu + J_{new}^\mu = s u^\mu - \frac{u_\nu \pi^{\mu\nu}}{T} - \nu j^\mu + J_{new}^\mu. \end{aligned} \quad (3.39)$$

Substituting the field redefinition (2.18) into this equation and setting

$$\delta\phi(x^i) = h(x^i)$$

(recall  $h$  is a function only of space) we recover a new form of the stress tensor and currents

$$\begin{aligned} T^{\mu\nu} &= (\tilde{\epsilon} + \tilde{P})u^\mu u^\nu + \tilde{P} G^{\mu\nu} + \tilde{f} \tilde{\xi}^\mu \tilde{\xi}^\nu + \tilde{\pi}^{\mu\nu} \\ J^\mu &= \tilde{q} u^\mu - \tilde{f} \tilde{\xi}^\mu + \tilde{j}^\mu \\ J_s^\mu &= \tilde{J}_{can}^\mu + \tilde{J}_{new}^\mu = \tilde{s} u^\mu - \frac{u_\nu \tilde{\pi}^{\mu\nu}}{T} - \nu \tilde{j}^\mu + \tilde{J}_{new}^\mu \end{aligned} \quad (3.40)$$

with

$$\begin{aligned}
\tilde{\pi}^{\mu\nu} &= \pi^{\mu\nu} - \left[ \frac{\partial(\epsilon + P)}{\partial\chi} u^\mu u^\nu + \frac{\partial P}{\partial\chi} G^{\mu\nu} + \frac{\partial f}{\partial\chi} \xi^\mu \xi^\nu \right] (-2\xi \cdot \nabla^{(4)} h) \\
&\quad - f(\xi^\mu G^{\nu\alpha} \nabla_\alpha^{(4)} h + \xi^\nu G^{\mu\alpha} \nabla_\alpha^{(4)} h) \\
\tilde{j}^\mu &= j^\mu - \left[ \frac{\partial q}{\partial\chi} u^\mu - \frac{\partial f}{\partial\chi} \xi^\mu \right] (-2\xi \cdot \nabla^{(4)} h) + f G^{\mu\alpha} \nabla_\alpha^{(4)} h \\
\tilde{J}_{new}^\mu &= J_{new}^\mu - (-2\xi \cdot \nabla^{(4)} h) \left( \frac{\partial s}{\partial\chi} \right) u^\mu - \frac{u_\nu (\pi^{\mu\nu} - \tilde{\pi}^{\mu\nu})}{T} - \nu(j^\mu - \tilde{j}^\mu) \\
&= J_{new}^\mu + \frac{f}{T} (u^\mu \xi^\nu - u^\nu \xi^\mu) \nabla_\nu^{(4)} h = J_{new}^\mu + \frac{Q^{\mu\nu}}{T} \nabla_\nu^{(4)} h
\end{aligned} \tag{3.41}$$

All Greek indices in (3.41) and (3.40) run from  $1 \dots 4$  and are raised and lowered with the full four dimensional metric  $G^{\mu\nu}$ .  $\chi$  derivatives in (3.41) are taken at fixed  $T$  and  $\nu$ . In deriving last equality in (3.41) we have used the first law of thermodynamics.

$$d\epsilon = Tds + \nu dq - \frac{f}{2} d\chi$$

We will now independently verify that our final answers for  $J_{new}^\mu$  and the constitutive relations have this symmetry. To start with recall that, from (2.21) and (3.17),

$$\tilde{c}_a - c_a = \frac{1}{T} \frac{\partial h}{\partial H_a} \tag{3.42}$$

It follows immediately from (3.42) that the expression for  $J_{new}^\mu$

$$J_{new}^\mu = \sum_a c_a (\partial_\nu H_a) Q^{\mu\nu}$$

(see (3.3)) transforms as predicted by the last of (3.41).

We now turn to the verification that our results for transport coefficients, (3.11),(3.12), transform as predicted by (3.41). The algebra involved in a direct verification is formidable, so we will content ourselves with an indirect check. We first recall that we have already verified (see (3.15)) that

$$\begin{aligned}
\mathfrak{S}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{S}_a(\pi_{\mu\nu}, j_\mu) \\
\mathfrak{V}_a(\delta T_{\mu\nu}, \delta J_\mu) &= \mathfrak{V}_a(\pi_{\mu\nu}, j_\mu)
\end{aligned} \tag{3.43}$$

in fact this equation formed the basis of one of our two methods of determining constitutive relations. It follows that if we can show that  $\delta T_{\mu\nu}$  and  $\delta J_\mu$  obey (3.41), then the same will be true of (3.11),(3.12). (Recall  $\delta T_{\mu\nu}$  was the first order shift in the stress tensor arising from first order corrections to the Goldstone action;  $\delta J^\mu$  was similarly defined.) We will now check that this is indeed the case. In order to do this we first

simplify the (3.41) specializing to the case of stationary equilibrium

$$\begin{aligned}
(j_0 - \tilde{j}_0) &= -2e^\sigma \left[ \frac{\partial}{\partial \zeta_f^2} (q + \mu f) \right] (\zeta^{eq} \cdot \partial) h \\
&= e^\sigma \left[ \frac{\partial}{\partial \nu} \left( \frac{f}{T} \right) \right] (\zeta^{eq} \cdot \partial) h \\
(j^i - \tilde{j}^i) &= -f \nabla^i h - 2(\zeta^{eq} \cdot \partial h) \frac{\partial f}{\partial \zeta_f^2} (\zeta^{eq})^i
\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}
(\pi_{00} - \tilde{\pi}_{00}) &= -2e^{2\sigma} \left[ \frac{\partial(\epsilon + \mu^2 f)}{\partial \zeta_f^2} \right] (\zeta^{eq} \cdot \partial) h \\
&= 2e^{2\sigma} \left[ \frac{\partial}{\partial \zeta_f^2} \left( T \frac{\partial P}{\partial T} - P \right) \right] (\zeta^{eq} \cdot \partial) h \\
&= 2e^{2\sigma} \left( -\frac{T}{2} \frac{\partial f}{\partial T} + \frac{f}{2} \right) (\zeta^{eq} \cdot \partial) h \\
&= -T^2 e^{2\sigma} \left[ \frac{\partial}{\partial T} \left( \frac{f}{T} \right) \right] (\zeta^{eq} \cdot \partial) h \\
(\pi_0^i - \tilde{\pi}_0^i) &= -A_0 (j^i - \tilde{j}^i) \\
(\pi^{ij} - \tilde{\pi}^{ij}) &= 2(\zeta^{eq} \cdot \partial h) \left[ -\frac{f}{2} g^{ij} + \left( \frac{\partial f}{\partial \zeta_f^2} \right) (\zeta^{eq})^i (\zeta^{eq})^j \right] + f [(\zeta^{eq})^i \nabla^j h + (\zeta^{eq})^j \nabla^i h]
\end{aligned} \tag{3.45}$$

Where each of the scalar thermodynamic functions are evaluated on the zeroth order equilibrium solution

$$T = \hat{T}, \quad \nu = \hat{\nu}, \quad (\zeta_f)_i = \zeta_i^{eq}$$

In obtaining (3.44) and (3.45)

$$dP = \left( \frac{\epsilon + P + \mu^2 f}{T} \right) dT + T(q + \nu f) d\nu - \frac{f}{2} d\zeta_f^2$$

In those equations all spatial indices are raised and lowered by use of the spatial metric  $g_{ij}$  (all the free indices will run from 1 to 3).

We now turn to the explicit expressions for  $\delta T_{\mu\nu}$  and  $\delta J_\mu$  listed in (2.22). Substituting

$$\tilde{f}_3 = f_3 + h$$

(see (2.21)) in those expressions we obtain immediate agreement with (3.44) and (3.45). This completes our verification.



## 4. Constraints on parity violating constitutive relations at first order

In this section we use the partition function to derive constraints on parity violating contributions to constitutive relations by comparison with the local goldstone action (2.26). As in the previous subsection, we find perfect agreement with the constraints obtained from the local form of the second law. It turns out in this case that the second law analysis has already been performed, in full detail, in [6]. We begin this section by reviewing the results of [6], before turning to a re derivation of those results by comparison with (2.26).

### 4.1 Review of constraints from the second law

#### 4.1.1 Basis of Frame Invariants

As we have seen above, the constitutive relations are an expansion of frame invariant combinations of  $\pi^{\mu\nu}$  and  $j^\mu$  in terms of independent one derivative scalars, vectors and tensors. Before even specifying the constitutive relations, we must first specify a basis of frame invariant expressions that we will expand in this manner. In the previous section we choose to work with the frame invariant scalars  $\mathfrak{S}_a$  and frame invariant vectors  $\mathfrak{V}_a$ . A different choice for frame invariants was made in [6]; in order to ease comparison with the results of that paper, we will adapt that choice in this section. In this subsection we describe the basis of frame invariants used in [6].

Let

$$\begin{aligned}
\mathbf{s}_1 &= \pi^{\mu\nu} \tilde{P}_{\mu\nu} & \zeta_f^2 \mathbf{s}_2 &= \zeta_f \cdot \pi \cdot \zeta_f & (4.1) \\
\mathbf{s}_3 &= u \cdot \pi \cdot u & \mathbf{s}_4 &= u \cdot \pi \cdot \zeta_f \\
\mathbf{s}_5 &= u \cdot j & \mathbf{s}_6 &= \zeta_f \cdot j \\
\mathbf{s}_7 &= -\mu_{diss} \\
\mathbf{v}_1^\nu &= u_\mu \pi^{\mu\alpha} \tilde{P}^\nu_\alpha & \mathbf{v}_2^\nu &= (\zeta_f)_\mu \pi^{\mu\alpha} \tilde{P}^\nu_\alpha \\
\mathbf{v}_3^\nu &= \tilde{P}^\nu_\alpha j^\alpha \\
\mathbf{t} &= \tilde{P}_\mu{}^\alpha \tilde{P}_\nu{}^\beta \pi_{\alpha\beta} - \frac{1}{2} \tilde{P}^{\mu\nu} \tilde{P}^{\alpha\beta} \pi_{\alpha\beta} ,
\end{aligned}$$

Throughout this paper  $\mathbf{s}_7 = -\mu_{diss} = 0$  ( $\mu_{diss}$  was defined in [28]). However we will retain  $s_7$  in all our formulas, in order to permit easy adaptation of our final results to frames in which  $\mu_{diss} \neq 0$ .

$$P^{\mu\nu} = G^{\mu\nu} + u^\mu u^\nu \quad \tilde{P}^{\mu\nu} = P^{\mu\nu} - \frac{(\zeta_f)^\mu (\zeta_f)^\nu}{(\zeta_f)^2} . \quad (4.2)$$

Following [6] we define the row vectors

$$\begin{aligned}\mathbf{s} &= \left( \mathbf{s}_1 \ \mathbf{s}_2 \ \mathbf{s}_3 \ \mathbf{s}_4 \ \mathbf{s}_5 \ \mathbf{s}_6 \ \mathbf{s}_7 \right) \\ \mathbf{v} &= \left( \mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \right) .\end{aligned}\tag{4.3}$$

We also define the matrices

$$A^s = \begin{pmatrix} \frac{Rs}{2qT\psi_f} & \frac{B_3}{3T} - \frac{A_3}{2T\psi_f} & \frac{B_2}{3T} - \frac{A_2}{2T\psi_f} & \frac{B_1}{3T} - \frac{A_1}{2T\psi_f} \\ -\frac{Rs}{q\psi_f T} & \frac{B_3}{3T} + \frac{A_3}{T\psi_f} & \frac{B_2}{3T} + \frac{A_2}{T\psi_f} & \frac{B_1}{3T} + \frac{A_1}{T\psi_f} \\ 0 & \frac{1}{T^2} & 0 & 0 \\ -\frac{R}{T^2\psi_f} & \frac{K_3}{T} & \frac{K_2}{T} & \frac{K_1}{T} \\ 0 & 0 & -1 & 0 \\ -\frac{1}{T^2\psi_f} & 0 & 0 & 0 \\ 0 & \frac{(\rho+P)K_3}{T} & \frac{(\rho+P)K_2}{T} & \frac{(\rho+P)K_1}{T} \end{pmatrix}, \quad A^v = \begin{pmatrix} -R & 0 \\ 0 & \frac{2}{T^3\psi_f} \\ -1 & 0 \end{pmatrix}\tag{4.4}$$

where

$$R = \frac{q}{\rho + P} \quad V^\mu = \frac{E^\mu}{T} - P^{\mu\nu} \partial_\nu \nu$$

and the  $A_i$ 's  $B_i$ 's,  $C_i$ 's and  $K_i$ 's defined as follows.

$$\begin{aligned}\nu &= \frac{\mu}{T}, \quad \psi_f = \frac{\zeta_f^2}{T^2}, \quad K = \frac{\nabla_\theta[f\xi^\theta]}{\epsilon + P}, \quad R = \frac{q}{\epsilon + P} \\ B_1 &= -\frac{\partial}{\partial\psi_f}[\log(s)], \quad B_2 = -\frac{\partial}{\partial\nu}[\log(s)], \quad B_3 = -\frac{\partial}{\partial T}[\log(s)] \\ K_1 &= \frac{s}{\epsilon + P} \frac{\partial}{\partial\psi_f} \left[ \frac{q}{s} \right], \quad K_2 = \frac{s}{\epsilon + P} \frac{\partial}{\partial\nu} \left[ \frac{q}{s} \right], \quad K_3 = \frac{s}{\epsilon + P} \frac{\partial}{\partial T} \left[ \frac{q}{s} \right] \\ A_1 &= -\frac{1}{2} - \nu\psi_f(1 - \mu R) \left[ \frac{\partial}{\partial\psi_f} \left( \frac{q}{s} \right) \right] + \frac{\psi_f}{3s} \frac{\partial s}{\partial\psi_f} \\ A_2 &= -\nu\psi_f(1 - \mu R) \left[ \frac{\partial}{\partial\nu} \left( \frac{q}{s} \right) \right] + \frac{\psi_f}{3s} \frac{\partial s}{\partial\nu} \\ A_3 &= -\nu\psi_f(1 - \mu R) \left[ \frac{\partial}{\partial T} \left( \frac{q}{s} \right) \right] + \frac{\psi_f}{3s} \left( \frac{\partial s}{\partial T} - \frac{3s}{T} \right) \\ V_\mu &= \frac{E_\mu}{T} - P_\mu^\sigma \nabla_\sigma \left[ \frac{\mu}{T} \right] \\ \Omega^\mu &= \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} u_\nu \nabla_\lambda (\zeta_f)_\sigma, \quad \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} u_\nu \nabla_\lambda u_\sigma, \quad B^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} u_\nu F_{\lambda\sigma} .\end{aligned}\tag{4.5}$$

In terms of (4.1)-(4.4), the frame invariant scalar, vector and tensor combinations of  $\pi^{\mu\nu}$ ,  $j^\mu$  and  $\mu_{diss}$  are given by the row vectors

$$\mathbf{s}A^s, \quad \mathbf{v}_\mu A^v, \quad \mathbf{t}_{\mu\nu} .\tag{4.6}$$

By scalars, vectors and tensors we mean expressions which transform as spin 0,  $\pm 1$  and  $\pm 2$  representations of the  $SO(2)$  symmetry that is left invariant by the two vectors  $u^\mu$  and  $\xi^\mu$  at each point in spacetime.

vector	definition	dual parity odd vector ( $\epsilon^{\mu\nu\alpha\beta}u_\nu\xi_\alpha\mathcal{V}_\beta$ ) evaluated in equilibrium
$\mathcal{V}_1^\mu$	$(\frac{E^\mu}{T} - \nabla^\mu(\frac{\mu}{T}))$	0
$\mathcal{V}_2^\mu$	$\tilde{P}^{\mu\beta}(\zeta_f^\alpha\sigma_{\alpha\beta})$	0
$\mathcal{V}_3^\mu$	$\tilde{P}^{\mu\sigma}\nabla_\sigma T$	$-\hat{T}V_1^i$
$\mathcal{V}_4^\mu$	$\tilde{P}^{\mu\sigma}\nabla_\sigma(\frac{\mu}{T})$	$\frac{1}{T_0}V_2^i$
$\mathcal{V}_5^\mu$	$\tilde{P}^{\mu\sigma}\nabla_\sigma(\frac{\zeta_f^2}{T^2})$	$\epsilon^{ijk}V_5^i$
$\mathcal{V}_6^\mu$	$\frac{\mathcal{V}_2^{e\mu} - \tilde{P}^{\mu\alpha}\zeta_f^\nu\partial_\alpha u_\nu}{\zeta_f^2}$	$\frac{1}{2(\zeta^{eq})^2}e^\sigma V_4^i$
$\mathcal{V}_7^\mu$	$-\frac{P^{\mu\sigma}F_{\nu\alpha}\zeta_f^\alpha}{\zeta_f^2}$	$-\frac{1}{(\zeta^{eq})^2}(\xi_0 V_4^i + V_3^i)$

**Table 1:** Independent fluid vector data. Here  $V_m^i$  for  $m=1,2,3,4,5$  are independent vectors in equilibrium defined in (2.28)

pseudo scalars	definition	In equilibrium
$\tilde{\mathcal{S}}_1$	$\omega.\xi$	$-\frac{1}{2}e^\sigma S_2$
$\tilde{\mathcal{S}}_2$	$B.\xi$	$S_1 + \xi_0 S_2$
pseudo tensors	definition	In equilibrium
$\tilde{\mathcal{T}}_1^{\mu\nu}$	$*\sigma_{\mu\nu}^u$	0
$\tilde{\mathcal{T}}_2^{\mu\nu}$	$*\sigma_{\mu\nu}^\xi$	won't need

**Table 2:** Independent fluid scalar and tensor data. Here  $S_m$  for  $m=1,2$  are independent vectors in equilibrium defined in (2.28).  $\sigma^u$  and  $\sigma^\xi$  are the shear tensors for normal and superfluid velocity respectively and  $*\sigma_{\mu\nu} = \epsilon^{\mu\rho\alpha\beta}u^\rho\xi^\alpha\sigma_\beta^\nu + (\mu \leftrightarrow \nu)$

#### 4.1.2 Constitutive Relations

We have 4 frame invariant scalars, 2 frame invariant vectors and one frame invariant tensor. The most general symmetry allowed parity odd first derivative constitutive relations take the form

$$\begin{aligned}
\mathbf{t}^{\mu\nu} &= -\tilde{\eta}\tilde{\mathcal{T}}_1^{\mu\nu} \\
\mathbf{v}_i^\mu A_{ij}^v &= -\sum_{i=1}^2 \tilde{\mathcal{V}}_i \tilde{\kappa}_{ij} - \left( \sum_{i=3}^7 \tilde{\mathcal{V}}_i \tilde{\kappa}_{ij} \right) \\
\mathbf{s}_i A_{ij}^s &= -\left( \sum_{i=1}^2 \sum_{j=1}^4 \tilde{\mathcal{S}}_i \tilde{\beta}_{ij} \right)
\end{aligned} \tag{4.7}$$

with  $\mathcal{T}$ ,  $\tilde{\mathcal{T}}$ ,  $\mathcal{V}$ ,  $\tilde{\mathcal{V}}$ ,  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  a basis of onshell independent  $SO(2)$  invariant tensors, vectors and scalars given in table

### 4.1.3 Constraints on constitutive relations from the local second law

Notice that both pseudo tensors that appear in (4.7) are nondissipative. Further, the five pseudo vectors  $\mathcal{V}_i$   $i = 3 \dots 7$ , are nondissipative. It may therefore come as no surprise to the reader that [6] was able to use the principle of local entropy increase to determine  $\tilde{\kappa}_{im}$  ( $i = 3 \dots 7$  and  $m = 1 \dots 2$ ), together with  $\tilde{\beta}_{ij}$  ( $i = 1 \dots 2$  and  $j = 1 \dots 4$ ) in terms of two free functions that appeared in the parameterization of the entropy current. These two functions were called  $\sigma_8$  and  $\sigma_{10}$  in [6]. The results of [6] were presented in terms of  $\sigma_8$  and  $\sigma_{10}$  and four additional auxiliary fields which were determined in terms of  $\sigma_8$  and  $\sigma_{10}$  by the relations <sup>17</sup>

$$\begin{aligned}\sigma_3 &= -T \frac{\partial}{\partial T} (\sigma_{10} - \nu \sigma_8) \\ \sigma_4 &= \sigma_8 + C\nu + 2\tilde{h} - \frac{\partial}{\partial \nu} (\sigma_{10} - \nu \sigma_8) \\ \sigma_5 &= -\frac{\partial}{\partial \psi_f} (\sigma_{10} - \nu \sigma_8) \\ \sigma_9 &= 2\nu(\sigma_{10} - \nu \sigma_8) - \frac{2}{3}C\nu^3 - 2\tilde{h}\nu^2 + s_9\end{aligned}\tag{4.8}$$

In terms of all these fields, it was demonstrated in [6] that point wise positivity of the the divergence of the entropy current determines

$$\tilde{\eta} = 0, \quad \tilde{\kappa}_{m2} = 0\tag{4.9}$$

and

$$\begin{aligned}\tilde{\kappa}_{31} &= -RT\sigma_3 - T\partial_T\sigma_8 \\ \tilde{\kappa}_{41} &= -RT^2\sigma_4 - T\partial_\nu\sigma_8 \\ \tilde{\kappa}_{51} &= -RT^2\sigma_5 - T\partial_{\psi_f}\sigma_8 \\ \tilde{\kappa}_{61} &= -2RT^3\sigma_9 + 2T^2\sigma_{10} \\ \tilde{\kappa}_{71} &= -RT^2\sigma_{10} + 2T\sigma_8 + CT\nu + 2\tilde{h}T\end{aligned}\tag{4.10}$$

$$-\tilde{\beta}_{ij} = \begin{pmatrix} \frac{2RT\sigma_9}{\psi_f} - \frac{2\sigma_{10}}{\psi_f} & -2\sigma_3 - 2T^2K_3\sigma_9 & -2T\sigma_4 - 2T^2K_2\sigma_9 & -2T\sigma_5 - 2K_1T^2\sigma_9 \\ -\frac{C\nu+2\tilde{h}}{T\psi_f} - \frac{2\sigma_8}{T\psi_f} + \frac{R\sigma_{10}}{\psi_f} & \partial_T\sigma_8 - K_3T\sigma_{10} & \partial_\nu\sigma_8 - K_2T\sigma_{10} & \partial_{\psi_f}\sigma_8 - K_1T\sigma_{10} \end{pmatrix}.\tag{4.11}$$

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<sup>17</sup>All terms in (4.8) proportional to the constant  $\tilde{h}$  were omitted in [6]. The reason for this is that [6] assumed that the entropy current was gauge invariant. As explained in [12] this does not seem to be physically necessary as long as the divergence of the entropy current is gauge invariant. This allows the addition of the new term proportional to  $\tilde{h}$  in (4.21), which allows for a slight modification of the results of [6], captured by the shifts described below. As we will see later, the requirement of CPT invariance forces  $\tilde{h}$  to vanish.

## 4.2 Constraints on constitutive relations from the Goldstone action

As in the previous section, we use the Goldstone action to constrain transport coefficients as follows. All constraints follow from the analogue of (3.15)

$$\begin{aligned} \mathbf{t}^{\mu\nu}(\delta T_{\mu\nu}, \delta J_\mu) &= \mathbf{t}^{\mu\nu}(\pi_{\mu\nu}, j_\mu) \\ \mathbf{v}_i^\mu A_{ij}^v(\delta T_{\mu\nu}, \delta J_\mu) &= \mathbf{v}_i^\mu A_{ij}^v(\pi_{\mu\nu}, j_\mu) \\ \mathbf{s}_i A_{ij}^s(\delta T_{\mu\nu}, \delta J_\mu) &= \mathbf{s}_i A_{ij}^s(\pi_{\mu\nu}, j_\mu) \end{aligned} \quad (4.12)$$

The LHS in this equation may be determined in terms of the functions  $g_1$  and  $g_2$  in the Goldstone action using (2.28). The RHS of the same equation is simplified using (4.7) under the substitution  $T \rightarrow \hat{T}$ ,  $\mu \rightarrow \hat{\mu}$ ,  $\zeta_f \rightarrow \zeta^{eq}$ . Under the last substitution, the parity odd first derivative vectors and scalars evaluate to geometric expressions. Substituting these relations into the RHS of (4.12) and equating coefficients of independent vectors and tensors yields an expression for all non dissipative transport coefficients in terms of the functions  $g_1$  and  $g_2$ . Using Eq.(2.27) one obtains

$$\begin{aligned} \mathbf{v}_1^i &= u_\mu \pi^{\mu\alpha} \tilde{P}_\alpha^i \\ &= \hat{T}^3(-\hat{\nu} \partial_{\hat{T}} g_1 + \partial_{\hat{T}} g_2) V_1^i + \frac{\hat{T}^2}{T_0}(\hat{\nu} \partial_{\hat{\nu}} g_1 - \partial_{\hat{\nu}} g_2) V_2^i - \frac{\hat{T}^2}{(\zeta^{eq})^2}(g_2 - 2g_1 \hat{\nu}) V_3^i \\ &\quad + T_0 \hat{\nu} \frac{\hat{T}^2}{(\zeta^{eq})^2} V_4^i + \hat{T}^2(\hat{\nu} \partial_{\psi_{eq}} g_1 - \partial_{\psi_{eq}} g_2) V_5^i \\ \mathbf{v}_2^i &= (\zeta_f)_\mu \pi^{\mu\alpha} \tilde{P}_\alpha^i = 0 \\ \mathbf{v}_3^i &= \tilde{P}_\alpha^\nu j^\alpha \\ &= \hat{T}(\hat{T} \partial_{\hat{T}} g_1 V_1^i - \frac{1}{T_0} \partial_{\hat{\nu}} g_1 V_2^i - \frac{1}{(\zeta^{eq})^2}(2g_1 V_3^i + g_2 T_0 V_4^i + (\zeta^{eq})^2 \partial_{\psi_{eq}} g_1 V_5^i)) \\ \mathbf{s}_1 &= \pi^{\mu\nu} \tilde{P}_{\mu\nu} = 0 \\ \mathbf{s}_2 &= \frac{1}{(\zeta_f)^2} \zeta_f \cdot \pi \cdot \zeta_f = -\frac{2}{\hat{T}}(\zeta^{eq})^2(\partial_{\psi_{eq}} g_1 S_1 + T_0 \partial_{\psi_{eq}} g_2 S_2) \\ \mathbf{s}_3 &= u \cdot \pi \cdot u = \hat{T}(\hat{T} \partial_{\hat{T}} g_1 - 2\psi_{eq} \partial_{\psi_{eq}} g_1) S_1 + \hat{T} T_0(\hat{T} \partial_{\hat{T}} g_2 - 2\psi_{eq} \partial_{\psi_{eq}} g_2) S_2 \\ \mathbf{s}_4 &= u \cdot \pi \cdot \zeta_f = (\hat{T}^2(g_2 - 2g_1) - 2(\zeta^{eq})^2 \hat{\nu} \partial_{\psi_{eq}} g_1) S_1 - S_2 T_0 \hat{\nu} (g_2 \hat{T}^2 + 2(\zeta^{eq})^2 \partial_{\psi_{eq}} g_2) \\ &\quad + 2C_1 e^{-\sigma} S_2 T_0^3 - C \frac{1}{6} A_0^2 e^{-\sigma} (A_0 S_2 + 3S_1) \\ \mathbf{s}_5 &= u \cdot j = -(\partial_{\hat{\nu}} g_1 S_1 + T_0 \partial_{\hat{\nu}} g_2 S_2) \\ \mathbf{s}_6 &= \zeta_f \cdot j = 2\hat{T}(g_1 + \frac{(\zeta^{eq})^2}{\hat{T}^2} \partial_{\psi_{eq}} g_1) S_1 + \hat{T} T_0(g_2 + \frac{(\zeta^{eq})^2}{\hat{T}^2} \partial_{\psi_{eq}} g_2) S_2 + \frac{1}{2} C A_0 e^{-\sigma} (A_0 S_2 + 2S_1) \\ \mathbf{s}_7 &= -\mu_{diss} = 0 \end{aligned} \quad (4.13)$$

Now using Eq.(4.7) one can find out the transport coefficients  $\tilde{\kappa}_{ij}$  in terms of partition

function coefficients  $g_1, g_2$  as follows

$$\begin{aligned}
\tilde{\eta} &= 0, \quad \tilde{\kappa}_{i2} = 0 \quad \text{for } i \in (3 \text{ to } 7) \\
\kappa_{31} &= -\frac{\hat{T} \left( (-\hat{\nu}q\hat{T} + \epsilon + P)\partial_{\hat{T}}g_1 + q\hat{T}\partial_{\hat{T}}g_2 \right)}{P + \epsilon} \\
\kappa_{41} &= -\frac{\hat{T} \left( (-\hat{\nu}q\hat{T} + \epsilon + P)\partial_{\hat{\nu}}g_1 + q\hat{T}\partial_{\hat{\nu}}g_2 \right)}{P + \epsilon} \\
\kappa_{51} &= -\frac{\hat{T} \left( (-\hat{\nu}q\hat{T} + \epsilon + P)\partial_{\psi_{eq}}g_1 + q\hat{T}\partial_{\psi_{eq}}g_2 \right)}{P + \epsilon} \\
\kappa_{61} &= \frac{2\hat{T}^2}{\epsilon + P} \left( -g_2(-2\hat{\nu}q\hat{T} + \epsilon + P) + 2g_1\hat{\nu}(-\hat{\nu}q\hat{T} + \epsilon + P) \right) \\
&\quad + \frac{C\hat{\nu}^2\hat{T}^2(3p - 2\hat{\nu}q\hat{T} + 3\epsilon)}{3(P + \epsilon)} - \frac{4C_1q\hat{T}^3}{P + \epsilon} \\
\kappa_{71} &= \frac{\hat{T}}{\epsilon + P} \left( g_2q\hat{T} + 2g_1(-\hat{\nu}q\hat{T} + \epsilon + P) \right) + \frac{C\hat{\nu}\hat{T}(2p - \hat{\nu}q\hat{T} + 2\epsilon)}{2(P + \epsilon)}.
\end{aligned} \tag{4.14}$$

Similarly, the transport coefficients  $\beta_{ij}$  in terms of partition function coefficients  $g_1, g_2$  as follows

$$\begin{aligned}
-\beta_{11} &= \frac{4R\hat{T}\hat{\nu}(-g_2 + g_1\hat{\nu})}{\psi_{eq}} - \frac{2(-g_2 + 2g_1\hat{\nu})}{\psi_{eq}} + C \frac{\hat{\nu}^2\hat{T}^2(-3P + 2\hat{\nu}q\hat{T} - 3\epsilon)}{3(\zeta^{eq})^2(P + \epsilon)} + C_1 \frac{4q\hat{T}^3}{(\zeta^{eq})^2(P + \epsilon)} \\
-\beta_{12} &= -\frac{2g_1}{\hat{T}\psi_{eq}} + \frac{R(-g_2 + 2g_1\hat{\nu})}{\psi_{eq}} - C \frac{\hat{\nu}\hat{T}(2P - \hat{\nu}q\hat{T} + 2\epsilon)}{2(\zeta^{eq})^2(P + \epsilon)} \\
-\beta_{21} &= -2\hat{T}(-\hat{\nu}\partial_{\hat{T}}g_1 + \partial_{\hat{T}}g_2) - 4\hat{\nu}T^2K_3(-g_2 + g_1\hat{\nu}) - \frac{2}{3}CK_3\hat{T}^2\hat{\nu}^3 - 4C_1K_3\hat{T}^2 \\
-\beta_{22} &= \partial_{\hat{T}}g_1 - K_3\hat{T}(-g_2 + 2g_1\hat{\nu}) - \frac{1}{2}CK_3\hat{T}\hat{\nu}^2 \\
-\beta_{31} &= -2\hat{T}(-\hat{\nu}\partial_{\hat{\nu}}g_1 + \partial_{\hat{\nu}}g_2) - 4\hat{\nu}\hat{T}^2K_2(-g_2 + g_1\hat{\nu}) - \frac{2}{3}CK_2\hat{T}^2\hat{\nu}^3 - 4C_1K_2\hat{T}^2 \\
-\beta_{32} &= \partial_{\hat{\nu}}g_1 - K_2\hat{T}(-g_2 + 2g_1\hat{\nu}) - \frac{1}{2}CK_2\hat{T}\hat{\nu}^2 \\
-\beta_{41} &= -2\hat{T}(-\hat{\nu}\partial_{\psi_{eq}}g_1 + \partial_{\psi_{eq}}g_2) - 4K_1\hat{\nu}\hat{T}^2(-g_2 + g_1\hat{\nu}) - \frac{2}{3}CK_1\hat{T}^2\hat{\nu}^3 - 4C_1K_1\hat{T}^2, \\
-\beta_{42} &= \partial_{\psi_{eq}}g_1 - K_1\hat{T}(-g_2 + 2g_1\hat{\nu}) - \frac{1}{2}CK_1\hat{T}\hat{\nu}^2.
\end{aligned} \tag{4.15}$$

In equations (4.13), (4.14) and (4.15) the functions  $g_1, g_2$  and all the other thermodynamics functions (like  $\epsilon, P, q$  etc) as arbitrary functions of  $\hat{T}, \hat{\nu}$  and  $\psi_{eq}$ .

If we make the substitution

$$g_1 = \sigma_8 + \tilde{h}, \quad g_2 = -\sigma_{10} + 2\hat{\nu}\sigma_8 + \frac{1}{2}C\hat{\nu}^2 + 2\tilde{h}\hat{\nu}. \tag{4.16}$$

and introduce the auxiliary fields  $\sigma_3, \sigma_4, \sigma_5$  and  $\sigma_9$  which are written in terms of  $\sigma_8$  and  $\sigma_{10}$  in (4.8) then our results for nondissipative transport coefficients agree precisely<sup>18</sup> with (4.9), (4.10), (4.11).

### 4.3 Entropy

As in the parity even case, we may determine the parity odd contribution to the entropy current by a simple direct comparison with the the partition function. The relevant equation here is

$$\begin{aligned} W_1^{odd} + W_{anom} &= \int d^3y \sqrt{-G} [\hat{\nu}(\delta J_{consistent}^0 - \delta J_{covariant}^0) + J_{S_{new}}^0] \\ &= \int d^3y \sqrt{-G} [\hat{\nu} \delta J_{shift}^0 + J_{S_{new}}^0] \end{aligned} \quad (4.17)$$

The term in (4.17) proportional to  $\delta J_{shift}^0$  has its origin in the fact that (3.27) is correct when  $J^0$  is taken to be the consistent  $U(1)$  current. On the other hand the canonical entropy current of hydrodynamics is defined in terms of the covariant  $U(1)$  current. As explained in [12] these two currents differ by the shift

$$j_{shift}^\mu = \frac{C}{6} \epsilon^{\mu\nu\rho\sigma} \mathcal{A}_\nu \mathcal{F}_{\rho\sigma}. \quad (4.18)$$

The contribution of this shift to the RHS of (4.17) evaluates to

$$\begin{aligned} & \int d^3y \sqrt{-G} \hat{\nu} \delta J_{shift}^0 \\ &= \frac{C}{6} \int d^3y \sqrt{-G} e^{-\sigma} \hat{\nu} \epsilon^{ijk} \mathcal{A}_i \mathcal{F}_{jk} \\ &= \frac{C}{3} \int d^3y \sqrt{g} \hat{\nu} \epsilon^{ijk} \left( A_i \partial_j A_k + A_0 A_i \partial_j a_k - A_i a_j \partial_k A_0 + A_0 a_i \partial_j A_k + A_0^2 a_i \partial_j a_k \right) \\ &= \frac{C}{3} \int d^3y \sqrt{g} \hat{\nu} \epsilon^{ijk} \left( A_i \partial_j A_k + \frac{1}{2} A_0 A_i \partial_j a_k + \frac{3}{2} A_0 a_i \partial_j A_k + A_0^2 a_i \partial_j a_k \right) \\ &= \frac{C}{3} \int d^3y \sqrt{g} \hat{\nu} \epsilon^{ijk} \left( A_i \partial_j A_k + \frac{1}{2} A_0 A_i \partial_j a_k + \frac{3}{2 \hat{T}^2 \psi_{eq}} A_0 (a \cdot (\zeta^{eq}) S_1 - a \cdot V_3) \right. \\ & \quad \left. + \frac{A_0^2}{\hat{T}^2 \psi_{eq}} (a \cdot (\zeta^{eq}) S_2 - a \cdot V_4) \right) \end{aligned} \quad (4.19)$$

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<sup>18</sup>We also need to make identification  $s_9 = 2C_1$ , as will be clear below.

Taking this contribution to the LHS of eq.(4.17) we find

$$\begin{aligned}
& \int d^3y \sqrt{-G} J_{S_{\text{new}}}^0 \\
&= W_1^{\text{odd}} + W_{\text{anom}} - \int d^3y \sqrt{-G} \delta J_{\text{shift}}^0 \\
&= \int d^3y \sqrt{g} \left( g_1 S_1 + T_0 g_2 S_2 + \frac{1}{\hat{T}^2 \psi_{eq}} (C_1 T_0^2 - \frac{C}{3} \hat{\nu} A_0^2) (a.(\zeta^{eq}) S_2 - a.V_4) \right. \\
&\quad \left. - \frac{1}{2 \hat{T}^2 \psi_{eq}} C \hat{\nu} A_0 (a.(\zeta^{eq}) S_1 - a.V_3) \right)
\end{aligned} \tag{4.20}$$

In rest of the section we will use (4.20) to constrain the new part of the entropy current. The most general form of the first order entropy current is given by

$$\begin{aligned}
J_{S_{\text{new}}}^\mu &= \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\sigma_1 T u_\rho \zeta_\sigma) + \sigma_3 \tilde{\mathcal{V}}_3^{c\mu} + T \sigma_4 \tilde{\mathcal{V}}_4^{c\mu} + T \sigma_5 \tilde{\mathcal{V}}_5^{c\mu} \\
&\quad + \frac{\sigma_8}{2} \epsilon^{\mu\nu\rho\sigma} \xi_\nu F_{\rho\sigma} + T^2 \sigma_9 \omega^\mu + T \sigma_{10} B^\mu \\
&\quad + \alpha_1 \tilde{\mathcal{V}}_1^{c\mu} + \alpha_2 \tilde{\mathcal{V}}_2^{c\mu} + \zeta_f^\mu [\alpha_3 (\omega \cdot \zeta) + \alpha_4 (B \cdot \zeta)] + \tilde{h} \epsilon^{\mu\nu\lambda\sigma} \mathcal{A}_\nu \partial_\lambda \mathcal{A}_\sigma
\end{aligned}$$

where  $\tilde{h}$  is a constant (4.21)

Since the first term proportional to  $\sigma_1$  is a total derivative, it is not determined. The term proportional to  $\alpha_1$  and  $\alpha_2$  is also undetermined as  $\tilde{\mathcal{V}}_1^{c\mu}$  and  $\tilde{\mathcal{V}}_2^{c\mu}$  both are zero at equilibrium. We now evaluate (4.21) in equilibrium. Using Table 1 and Table 2 and

$$\begin{aligned}
\tilde{\mathcal{V}}_0^{cI} &= 0, \quad \tilde{\mathcal{V}}^{c0,I} = -a_i \tilde{\mathcal{V}}^{ci,I} \quad \text{where } I \in (1 \text{ to } 7) \\
\omega^0 &= \frac{e^\sigma}{2(\zeta^{eq})^2} ((a.(\zeta^{eq})) S_2 - (a.V_4)), \\
B^0 &= -\frac{1}{(\zeta^{eq})^2} ((a.(\zeta^{eq})) S_1 - (a.V_3)) - \frac{A_0}{(\zeta^{eq})^2} ((a.(\zeta^{eq})) S_2 - (a.V_4)) \\
\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \xi_\nu F_{\rho\sigma} &= e^\sigma (S_1 + A_0 S_2) + \frac{A_0 e^\sigma}{(\zeta^{eq})^2} ((a.(\zeta^{eq})) S_1 - (a.V_3)) + \frac{A_0^2 e^\sigma}{(\zeta^{eq})^2} ((a.(\zeta^{eq})) S_2 - (a.V_4)) \\
&\quad + e^\sigma (a.V_2) \\
\epsilon^{0\nu\lambda\sigma} \mathcal{A}_\nu \partial_\lambda \mathcal{A}_\sigma &= e^{-\sigma} \epsilon^{ijk} [A_i \partial_j A_k + 2T_0 \hat{\nu} a_i \partial_j A_k + T_0^2 \hat{\nu}^2 a_i \partial_j a_k + \partial_i (T_0 \hat{\nu} a_j A_k)]
\end{aligned} \tag{4.22}$$



where  $V_i$  and  $S_i$  are listed in Eq.2.28. Now using the fact that  $(\zeta^{eq})_i = A_i + \partial_i \phi$

$$\begin{aligned}
& \int \sqrt{-G} \epsilon^{0\nu\lambda\sigma} \mathcal{A}_\nu \partial_\lambda \mathcal{A}_\sigma \\
&= \int \epsilon^{ijk} [A_i \partial_j A_k + 2T_0 \hat{\nu} a_i \partial_j A_k + T_0^2 \hat{\nu}^2 a_i \partial_j a_k] \\
&= \int \sqrt{g} \epsilon^{ijk} [(\zeta^{eq})_i \partial_j (\zeta^{eq})_k + 2T_0 \hat{\nu} a_i \partial_j (\zeta^{eq})_k + T_0^2 \hat{\nu}^2 a_i \partial_j a_k] \\
&= \int \sqrt{g} (S_1 + \frac{1}{(\zeta^{eq})^2} 2T_0 \hat{\nu} ((a.(\zeta^{eq}))S_1 - a.V_3) + \frac{1}{(\zeta^{eq})^2} T_0^2 \hat{\nu}^2 ((a.(\zeta^{eq}))S_2 - a.V_4))
\end{aligned} \tag{4.23}$$

we obtain

$$\begin{aligned}
\int d^3x \sqrt{-G} J_{S_{\text{new}}}^0 &= \int d^3x \sqrt{g} \left( e^\sigma \hat{T} \sigma_3(a.V_1) + (\sigma_8 - \sigma_4)(a.V_2) - \hat{T} e^\sigma \sigma_5(a.V_5) \right. \\
&\quad + \sigma_8(S_1 + A_0 S_2) + \tilde{h} S_1 - (a.(\zeta^{eq})) \left( -\frac{1}{2} \alpha_3 e^\sigma S_2 + \alpha_4 (S_1 + A_0 S_2) \right) \\
&\quad + \frac{1}{(\zeta^{eq})^2} (-\hat{T} e^\sigma \sigma_{10} + \sigma_8 A_0 + 2\tilde{h} T_0 \hat{\nu}) ((a.(\zeta^{eq}))S_1 - (a.V_3)) \\
&\quad \left. + \frac{1}{(\zeta^{eq})^2} \left( \frac{e^{2\sigma}}{2} \hat{T}^2 \sigma_9 - \hat{T} e^\sigma \sigma_{10} A_0 + \sigma_8 A_0^2 + \tilde{h} T_0^2 \hat{\nu}^2 \right) ((a.(\zeta^{eq}))S_2 - (a.V_4)) \right)
\end{aligned} \tag{4.24}$$

It is convenient to introduce the following redefinitions

$$\sigma_3 = -\hat{T} \partial_{\hat{T}} X, \quad \sigma_8 - \sigma_4 = \partial_{\hat{\nu}} X + Y, \quad \sigma_5 = -\partial_{\psi_{eq}} X + Z. \tag{4.25}$$

Now using

$$\partial_k X = \partial_{\hat{T}} X \partial_k \hat{T} + \partial_{\hat{\nu}} X \partial_k \hat{\nu} + \partial_{\psi_{eq}} X \partial_k \psi_{eq}, \tag{4.26}$$

the first line of the Eq.4.24 can be rewritten as

$$\begin{aligned}
& \int d^3x \sqrt{g} \left( e^\sigma \hat{T} \sigma_3(a.V_1) + (\sigma_8 - \sigma_4)(a.V_2) - \hat{T} e^\sigma \sigma_5(a.V_5) \right) \\
&= \int d^3x \sqrt{g} \left( T_0 \epsilon^{ijk} a_i (\zeta^{eq})_j \partial_k X + Y(a.V_2) - \hat{T} e^\sigma Z(a.V_5) \right) \\
&= \int d^3x \sqrt{g} \left( -T_0 X \epsilon^{ijk} (\zeta^{eq})_i \partial_j a_k + T_0 X \epsilon^{ijk} a_i \partial_j (\zeta^{eq})_k + Y(a.V_2) - \hat{T} e^\sigma Z(a.V_5) \right) \\
&= \int d^3x \sqrt{g} \left( -T_0 X S_2 + T_0 X \frac{1}{(\zeta^{eq})^2} ((a.(\zeta^{eq}))S_1 - (a.V_3)) + Y(a.V_2) - \hat{T} e^\sigma Z(a.V_5) \right).
\end{aligned} \tag{4.27}$$

So we obtain

$$\begin{aligned}
\int d^3x \sqrt{-G} J_{S_{\text{new}}}^0 &= \int d^3x \sqrt{g} \left( -T_0 X S_2 + \sigma_8 (S_1 + A_0 S_2) + \tilde{h} S_1 \right. \\
&\quad + (T_0 X - \hat{T} e^\sigma \sigma_{10} + \sigma_8 A_0 + 2\tilde{h} T_0 \nu) \frac{1}{(\zeta^{eq})^2} ((a.(\zeta^{eq})) S_1 - (a.V_3)) \\
&\quad + \frac{1}{(\zeta^{eq})^2} \left( \frac{e^{2\sigma}}{2} \hat{T}^2 \sigma_9 - \hat{T} e^\sigma \sigma_{10} A_0 + \sigma_8 A_0^2 + \tilde{h} T_0^2 \hat{\nu}^2 \right) ((a.(\zeta^{eq})) S_2 - (a.V_4)) \\
&\quad \left. - (a.(\zeta^{eq})) \left( -\frac{1}{2} \alpha_3 e^\sigma S_2 + \alpha_4 (S_1 + A_0 S_2) \right) + Y(a.V_2) - \hat{T} e^\sigma Z(a.V_5) \right).
\end{aligned} \tag{4.28}$$

Now using (4.20) we obtain

$$\begin{aligned}
Y &= Z = 0, \quad \alpha_3 = \alpha_4 = 0 \\
X &= \sigma_{10} - \hat{\nu} \sigma_8 - \frac{1}{2} C \hat{\nu}^2 - 2\tilde{h} \hat{\nu}, \\
\sigma_3 &= -\hat{T} \partial_{\hat{T}} (\sigma_{10} - \hat{\nu} \sigma_8), \quad \sigma_4 = \sigma_8 - \partial_{\hat{\nu}} (\sigma_{10} - \hat{\nu} \sigma_8) + C \hat{\nu} + 2\tilde{h}, \quad \sigma_5 = -\partial_{\psi_{eq}} (\sigma_{10} - \hat{\nu} \sigma_8) \\
\sigma_9 &= 2\hat{\nu} (\sigma_{10} - \hat{\nu} \sigma_8) + 2 \left( C_1 - \frac{C}{3} \hat{\nu}^3 \right) - 2\tilde{h} \hat{\nu}^2 \\
g_1 &= \sigma_8 + \tilde{h}, \quad g_2 = -\sigma_{10} + 2\hat{\nu} \sigma_8 + \frac{1}{2} C \hat{\nu}^2 + 2\tilde{h} \hat{\nu}.
\end{aligned} \tag{4.29}$$

<sup>19</sup> It may be verified that (4.29) is consistent with (4.8). In other words the entropy current determined by comparison with partition function agrees exactly with the non dissipative part of the entropy current determined from the requirement of positivity of divergence. <sup>20</sup>

## 5. CPT Invariance

In this section we explore the constraints imposed on the partition function (2.17) and (2.26) by the requirement of 4 dimensional CPT invariance. In Table 3 we list the action of CPT on various fields appearing in the partition function.

- **Parity even case:** Using this table one easily see that, demanding CPT invariance of the action (2.17), the functions  $f_1$ ,  $f_2$ ,  $f_3$  are even under CPT. Instead had we demanded only time reversal invariance, then the we would conclude that  $f_1 = f_2 = f_3 = 0$ .

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<sup>19</sup>The expression  $2C_1$  was referred to as  $s_9$  in [6].

<sup>20</sup>Note however that the entropy positivity method, in addition, determines two dissipative terms in the entropy current, and so, in that sense, carries more information about the entropy current.

Field	C	P	T	CPT
$\sigma$	+	+	+	+
$a_i$	+	-	-	+
$g_{ij}$	+	+	+	+
$A_0$	-	+	+	-
$A_i$	-	-	-	-
$\zeta_i$	-	-	-	-

**Table 3:** Action of CPT

- **Parity odd case:** Now demanding CPT invariance of the action (2.26), we conclude that  $g_1$  is odd function of  $A_0$  and hence it can not contain any constant. This in particular implies  $\tilde{h} = 0$ , since  $g_1 = \sigma_8 + \tilde{h}$ . So the gauge non invariant piece in entropy current in (4.21) vanishes once we demand CPT invariance. The function  $g_2$  appearing in (2.26) is even function in  $A_0$ . It is also easy to see that the requirement of CPT invariance of the partition function forces  $C_1 = 0$ .

## 6. Discussion

In this paper we have studied the equality type constraints between transport coefficients for relativistic superfluids at first order in the derivative expansion. Our central result is that the constraints obtained from a local form of the second law of thermodynamics agree exactly with those obtained from a study of the equilibrium partition function.

As the constraints obtained from both methods are numerous and rather involved in structure, the perfect agreement found in this paper strengthens the conjecture [12] that the constraints obtained from the partition function agree with those obtained from the local version of the second law of thermodynamics under all circumstances. It would be interesting to find either a proof for or a counterexample against this conjecture.

In the special case that the superfluid is nondissipative, [19] has presented a framework for describing superfluid dynamics from an action formalism. It would be interesting to understand the connection of the formalism of [19] to that described in this paper.

As we have explained above, a central object in our analysis was a local Euclidean action for the superconducting Goldstone field. In the neighborhood of a second order phase transition familiar Landau-Ginzburg action for the order parameter is the natural analogue of the Goldstone boson action utilized in this paper. It seems likely that the methods of the current paper generalize to the study of hydrodynamics in the

neighborhood of second order phase transitions (see [21] for a review). It would be interesting to perform this generalization.

Finally, in this paper we have discussed only the equality type constraints on nondissipative transport coefficients that follow from the local second law. We have neither discussed Onsager type equality constraints on dissipative coefficients nor the inequalities on dissipative coefficients. It is possible that these constraints follow the imposition of reasonable conditions (like stability) to time fluctuations about equilibrium. We leave the study of time dependence to future work.

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